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On the Stability of the Behavior of Random Walks on Groups

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ABSTRACT. We show that, for random walks on Cayley graphs, the long time behavior of the probability of return after $2n$ *steps is invariant by quasi-isometry.*

1. Introduction

Let G be a finitely generated group. For any finite generating set S satisfying $S = S^{-1}$, consider the Cayley graph (G, S) with vertex set G and an edge from x to y if and only if $y = xs$ for some $s \in S$. Thus, edges are oriented but this is merely a convention since (x, y) is an edge if and only if (y, x) is an edge. We allow the identity element id to be in S in which case our graph has a loop at each vertex. Clearly the graph (G, S) is invariant under the left action of G. Denote by $|x|$ the distance from the neutral element id to x in the Cayley graph (G, S), that is, $|x|$ is the minimal number k of elements of S needed to write x as $x = s_1 s_2 \dots s_k$, $s_i \in S$. The volume growth function of (G, S) is defined by

$$V(n) = \#\{x \in G : |x| \leq n\} .$$

This paper focuses on the probability of return after $2n$ steps of the simple random walk on (G, S). For a survey of this topic, see [36]. The simple random walk on (G, S) is the Markov process $(X_i)_0^\infty$ with values in G which evolves as follows: If the current state is x, the next state is a neighbor of x chosen uniformly at random. This implicitly defines a probability measure $\mathbf{P}_S$ on $G^{\mathbb{N}}$ such that

$$\mathbf{P}_S (X_n = y / X_0 = x) = \mu_S^{(n)} \left(x^{-1} y\right)$$

where

$$\mu_S(g) = \frac{1}{\# S} \mathbf{1}_S(g)$$

and $\mu^{(n)}$ is the n-fold convolution power of μ. Following usual notation we will also write $\mathbf{P}_S^x(\cdot) = \mathbf{P}_S(\cdot / X_0 = x)$ for the law of the walk based on S and started at $x \in G$. To avoid parity problems, we consider only the probability of return at even times and set

$$\phi_S(n) = \mathbf{P}_S^{\mathrm{id}} (X_{2n} = \mathrm{id}) = \mu_S^{(2n)}(\mathrm{id}) .$$

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It is well known and easy to see that $\phi_S(n) = \max_g \mu_S^{(2n)}(g)$, and it follows that ϕ_S is a non-increasing function.

We are interested in the behavior of $\phi_S(n)$ for large n and up to the equivalence relation $\simeq$ which we now define. Given two positive non-increasing functions u, v defined on the positive real axis, write $u \preceq v$ if there exist $C \geq 1$ such that

$$\forall\, t > 0, \quad u(t) \leq Cv(t/C)\,.$$

Write $u \simeq v$ if $u \preceq v$ and $v \preceq u$. When a function is defined only on the integers, we extend it to the positive real axis by linear interpolation. We will use the same name for the original function and its extension. In particular, we view ϕ_S as defined on the positive real axis.

The following theorem is the simplest result of this paper and illustrates well the type of questions that will be considered in the sequel.

Theorem 1.1. *Let G be a finitely generated group. Let S and T be two symmetric finite generating sets of G. Then*

$$\phi_S \simeq \phi_T\,.$$

To the best of our knowledge, this has not yet been proved in this generality. The result is not surprising. In fact, the idea that the behavior of ϕ_S does not depend on S is at the heart of Varopoulos' work in this area during the early 1980s [31, 32, 33, 34, 35]. This idea is also present in [2] where the invariance of the transient/recurrent character of random walks is treated. What is more surprising is that the relatively simple proof that we will give below has escaped notice until now. Actually, we will give two different proofs of Theorem 1.1.

Part of our interest in this result stems from the fact that there are many groups for which the behavior of ϕ_S is not known explicitly. This is the case for the groups of intermediate growth constructed by Grigorchuk [17]. Even among solvable finitely generated (or even finitely presented) groups, there are many for which the behavior of ϕ_S is not yet understood. See [27]. Still, for these groups, Theorem 1.1 says that the (unknown) behavior of the probability of return of a simple random walk after $2n$ steps is independent of the generating set.

Of course, there are several classes of groups for which Theorem 1.1 is known thanks to a detailed knowledge of the behavior of ϕ_S. This is the case for non-amenable groups since, by a theorem of Kesten [25],

$$\phi_S(n) \simeq \exp(-n)$$

for such groups. It is also the case for groups having polynomial growth of degree d (i.e., $V(n) \simeq (1+n)^d$), since Varopoulos [33] proved that

$$\phi_S(n) \simeq (1+n)^{-d/2}$$

in this case. See also [22, 35]. By celebrated results of Bass and Gromov, the groups of polynomial volume growth are exactly those groups that contain a nilpotent subgroup of finite index. Finally, groups that contain a polycyclic subgroup of finite index and have exponential volume growth satisfy

$$\phi_S(n) \simeq \exp\left(-n^{1/3}\right)\,.$$

See [1, 34, 22, 35]. Using these results and structure theorems one can prove Theorem 1.1 for all finitely generated groups that appear as discrete subgroup of a connected Lie group. Indeed, one can show that the behavior of ϕ_S for such a group must be of one of the following three "classical" types: $\exp(-n)$, $\exp(-n^{1/3})$, or $(1+n)^{-d/2}$ for some integer d. See [28, 26].

The importance of the notion of quasi-isometry in geometric group theory is well established. See, e.g., [19]. Section 4 generalizes Theorem 1.1 as follows.

Theorem 1.2. *Let (G, S) and (H, T) be Cayley graphs of two finitely generated groups G, H that are quasi-isometric. Then*

$$\phi_S \simeq \phi_T .$$

Many analytic properties are known to behave well under quasi-sometries, see, e.g., [24, 14] and the references therein. It is worth emphasizing that the technique we will use in the proof of Theorem 1.2 is similar but different and somewhat more refined than the techniques used in such references as [24, 14].

Another useful result concerns finitely generated subgroups of a finitely generated group.

Theorem 1.3. *Let (G, S) be a Cayley graph of the finitely generated group G. Let H be a finitely generated subgroup of G and T a finite symmetric generating set of H. Then*

$$\phi_S \preceq \phi_T .$$

Let μ be a probability measure on a countable group G. The random walk associated with μ is the Markov process $(X_i)_0^\infty$ with values in G which evolves as follows. If the current state is x, the next state is $y = xz$ where z is chosen at random according to μ. This notion generalizes that of simple random walk on a Cayley graph. We set

$$\phi_\mu(n) = \mathbf{P}_\mu\left(X_{2n} = \mathrm{id}/X_0 = \mathrm{id}\right) = \mathbf{P}_\mu^{\mathrm{id}}\left(X_{2n} = \mathrm{id}\right) = \mu^{(2n)}(\mathrm{id}) .$$

When μ is symmetric, that is, $\mu(x) = \mu(x^{-1})$ for all $x \in G$, then $\phi_\mu(n) = \max_g \mu^{(2n)}(g)$ and ϕ_μ is a non-increasing function. The next result generalizes both Theorem 1.1 and Theorem 1.3.

Theorem 1.4. *Let G be a finitely generated group with finite symmetric generating set S. Let μ be a symmetric probability measure on G such that μ has finite second moment, that is,*

$$\sum_{x \in G} |x|^2 \mu(x) < +\infty .$$

Then

$$\phi_S \preceq \phi_\mu .$$

The direct comparison in Theorem 1.4 allows us to obtain lower bounds on ϕ_μ if μ has finite second moment.

Corollary 1.5. *Let G be a finitely generated group with finite symmetric generating set S. Let μ be a probability measure on G such that*

$$\sum_{x \in G} |x|^2 \mu(x) < +\infty .$$

1. *If G has polynomial volume growth of degree d (that is $V(n) \simeq (1+n)^d$), then $\phi_\mu(n) \succeq (1+n)^{-d/2}$.*
2. *If G contains a polycyclic subgroup of finite index and has exponential volume growth, then $\phi_\mu(n) \succeq \exp\left(-n^{1/3}\right)$.*

Proof. The stated lower bounds are known if ϕ_μ is replaced by ϕ_S, S a finite symmetric generating set. See [1, 33, 22, 35]. Hence, the desired results follow readily from Theorem 1.4. Note that these lower bounds can be complemented with matching upper bounds if we assume in addition that the support of μ contains a finite generating set. See [35, Theorem VII.1.1]. □

Finally, in Section 5, Riemannian coverings of a compact manifold are considered. If M covers the compact manifold N with deck transformation group G, we show that the large time behavior of the heat kernel h_t on M is $\simeq$-equivalent to the behavior of simple random walk on G. This yields examples of Riemannian manifolds having new types of large time heat kernel behavior. See [27].

2. $\phi_S \simeq \phi_T$ and assorted results

This section establishes the $\simeq$-invariance of ϕ under changes of generating set, that is Theorem 1.1. In fact, we will prove Theorem 1.4, from which Theorems 1.1 and 1.3 follow. The purpose of this section is also to present the basic structure of proof and the main ideas that will later serve in Sections 4 and 5 to obtain Theorem 1.2 and the results concerning Riemannian coverings.

Lemma 2.1. *Let G be a finitely generated group, S a finite symmetric generating set. Let $|x|$ denote the distance between the neutral element id and x in the Cayley graph (G, S). Fix two symmetric probability measures μ_1, μ_2 on G and assume that*

$$c = \inf_{s\in S} \mu_2(s) > 0 \quad \text{and} \quad C = \sum_x |x|^2 \mu_1(x) < +\infty .$$

For f with finite support, consider

$$\mathcal{E}_i(f, f) = \frac{1}{2} \sum_{x,y} |f(x) - f(xy)|^2 \mu_i(y), \quad i = 1, 2 . \tag{2.1}$$

Then these Dirichlet forms satisfy

$$c\mathcal{E}_1 \le C\mathcal{E}_2 .$$

Proof. This is well known [33]. We give the proof for completness. If $\mu_1(y) > 0$, write $y = s_0 s_1 \cdots s_k$ with $s_0 = \mathrm{id}$, $s_i = s_i(y) \in S$, $i = 1, \dots, k$ and $k = |y|$. Then the Cauchy-Schwarz inequality yields

$$|f(x) - f(xy)|^2 \le k \sum_1^k |f(xs_0 \cdots s_{i-1}) - f(xs_0 \cdots s_i)|^2 .$$

Summing over x and using translation invariance of the counting measure, we get

$$\sum_x |f(x) - f(xy)|^2 \le k \sum_1^k \sum_x |f(x) - f(xs_i)|^2 .$$

Then

$$c \sum_x |f(x) - f(xy)|^2 \le 2|y|^2 \mathcal{E}_2(f, f) .$$

Multiplying by $\frac{1}{2}\mu_1(y)$ and summing over all $y \in G$ we obtain

$$c\mathcal{E}_1(f, f) \le C\mathcal{E}_2(f, f) .$$

This proves the lemma. □

For readers that are not familiar with the notion of Dirichlet form, let us recall that, for any symmetric probability measure μ on G, we have

$$\langle f * (\delta - \mu), f \rangle = \frac{1}{2} \sum_{x,y} |f(x) - f(xy)|^2 \mu(y)$$

where δ denotes the Dirac mass at the identity, $\delta(y) = \mathbf{1}_{\text{id}}(y)$. The form

$$\mathcal{E}_\mu(f, f) = \frac{1}{2} \sum_{x,y} |f(x) - f(xy)|^2 \mu(y)$$

on $\ell^2(G)$ is called the Dirichlet form of μ. In terms of operators, $\mathcal{E}_\mu$ is the quadratic form associated to the self-adjoint operator $(I - \mu^*)f = f * (\delta - \mu)$.

The following notation will be used several times throughout this paper. Let μ be a symmetric probability measure on G. Let A be a finite subset of G. Consider the kernel

$$K_{A,\mu}(x, y) = \begin{cases} \mu\left(x^{-1}y\right) & \text{if } x, y \in A \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

and the corresponding operator

$$K_{A,\mu} f(x) = \sum_y K_{A,\mu}(x, y) f(y)$$

acting on $\ell^2(A)$. This is a sub-Markovian operator, i.e.,

$$K_{A,\mu}(x, y) \geq 0 \text{ and } \sum_y K_{A,\mu}(x, y) \leq 1 .$$

When the measure μ under consideration is clear from the context, we will drop the direct reference to μ in this notation. Define the Dirichlet form of $K_{A,\mu}$ by setting

$$\mathcal{E}_{A,\mu}(f, f) = \left\langle \left(I - K_{A,\mu}\right) f, f \right\rangle$$

where the scalar product is in $\ell^2(A)$. Define the extension $\hat{f}$ of $f \in \ell^2(A)$ to G by setting $\hat{f}(x) = f(x)$ if $x \in A$ and $f(x) = 0$ otherwise. Then, for $x \in A$, $K_{A,\mu} f(x) = \hat{f} * \mu(x)$ and

$$\mathcal{E}_{A,\mu}(f, f) = \sum_{x \in A} \left(f(x) - K_{A,\mu} f(x)\right) f(x) = \mathcal{E}_\mu\left(\hat{f}, \hat{f}\right) . \tag{2.3}$$

Finally, the symmetry of μ implies that $K_{A,\mu}$ is a self-adjoint operator on $\ell^2(A)$ which is a finite dimensional Hilbert space. Thus, $K_{A,\mu}$ is diagonalizable with real eigenvalues. Denote its eigenvalues in non-increasing order by

$$\beta_{A,\mu}(\ell) \in [-1, 1], \quad \ell = 1, \ldots, \#A .$$

With this notation, the trace of $K^n_{A,\mu}$ is given by

$$\mathbf{Tr}\left(K^n_{A,\mu}\right) = \sum_x K^n_{A,\mu}(x, x) = \sum_1^{\#A} \beta_{A,\mu}(\ell)^n . \tag{2.4}$$

The following simple lemma is one of the keys of our proof of Theorems 1.1 and 1.3.

Lemma 2.2. *For $i = 1, 2$, let K_i be a symmetric sub-Markovian operator on a finite set A_i with associated Dirichet form $\mathcal{E}_i$. Let H_i be the finite dimensional Hilbert space of all real valued functions on A_i with the scalar product $\sum_{x\in A_i} f(x)g(x)$ and the norm $\|f\|_{i,2} = (\sum_{x\in A_i} |f(x)|^2)^{1/2}$. Assume that there exists a linear map $f \mapsto \tilde{f}$ from H_2 to H_1 such that, for all $f \in H_2$,*

$$\|f\|_{2,2} \le C_1 \left\|\tilde{f}\right\|_{1,2} \quad \text{and} \quad \mathcal{E}_1\left(\tilde{f}, \tilde{f}\right) \le C_2\mathcal{E}_2(f, f)\,.$$

Then

$$\mathbf{Tr}\left(K_2^{2n+1}\right) \le 2\left[\#A_2 e^{-n/B} + \mathbf{Tr}\left(K_1^{2\lfloor n/(2B)\rfloor}\right)\right]$$

with $B = C_1C_2$.

Proof. The hypotheses and the minimax characterization of eigenvalues (see, e.g., [21, p. 179]) implies that the eigenvalues of K_1, K_2 satisfy

$$1 - \beta_1(\ell) \le B\,(1 - \beta_2(\ell))\,, \quad \ell = 1, \ldots, \#A_2$$

(observe that $f \mapsto \tilde{f}$ is one to one so that $\#A_1 \ge \#A_2$). Hence, if $\beta_2(\ell) \ge 0$,

$$\beta_2(\ell)^{2n} \le \left[1 - \frac{1}{B}(1 - \beta_1(\ell))\right]^{2n} \le \exp\left(-\frac{2n}{B}(1 - \beta_1(\ell))\right)\,.$$

If $\beta_1(\ell) \ge 1/2$, it follows that

$$\beta_2(\ell)^{2n} \le \beta_1(\ell)^{n/B}$$

(use $x \ge e^{-2(1-x)}$ if $1/2 \le x \le 1$). Now, observe that

$$\mathbf{Tr}\left(K_2^{2n+1}\right) = \sum_x K_2^{2n+1}(x, x) = \sum_0^{\#A_2} \beta_2(\ell)^{2n+1} \ge 0\,.$$

Hence,

$$\sum_{\beta_2(\ell)<0} |\beta_2(\ell)|^{2n+2} \le \sum_{\beta_2(\ell)>0} \beta_2(\ell)^{2n}\,.$$

Finally,

$$\begin{aligned}
\mathbf{Tr}\left(K_2^{2(n+1)}\right) &\le 2\sum_{\beta_2(\ell)>0} \beta_2(\ell)^{2n} \\
&= 2\sum_{\substack{\beta_2(\ell)>0\\ \beta_1(\ell)<1/2}} \beta_2(\ell)^{2n} + 2\sum_{\substack{\beta_2(\ell)>0\\ \beta_1(\ell)\ge 1/2}} \beta_2(\ell)^{2n} \\
&\le 2\#A_2\, e^{-n/B} + 2\sum |\beta_1(\ell)|^{n/B} \\
&\le 2\left[\#A_2\, e^{-n/B} + \mathbf{Tr}\left(K_1^{2\lfloor n/(2B)\rfloor}\right)\right]\,.
\end{aligned}$$

□

Theorem 2.3. *Let G be a finitely generated group. Assume that G is amenable. Let μ_i, $i = 1, 2$, be two symmetric probability measures on G. Assume that the Dirichlet forms $\mathcal{E}_i$, $i = 1, 2$ defined at (2.1) satisfy $\mathcal{E}_1 \le B\mathcal{E}_2$. Then*

$$\mu_2^{(2n+2)}(id) \le 2\left[e^{-n/B} + \mu_1^{(2\lfloor n/(2B)\rfloor)}(id)\right]\,.$$

In particular,

$$\mu_2^{(2n)}(id) \preceq \mu_1^{(2n)}(id) .$$

Proof. Let A be a finite subset of G. Consider the kernels

$$K_{A,i} = K_{A,\mu_i}(x, y), \quad i = 1, 2$$

introduced at (2.2). For simplicity, we refer to the objects relative to μ_i by using the subscript $i \in \{1, 2\}$. The hypothesis $\mathcal{E}_1 \leq B\mathcal{E}_2$ implies $\mathcal{E}_{A,1} \leq B\mathcal{E}_{A,2}$ because of (2.3). Applying Lemma 2.2 with $A = A_1 = A_2$, $H_1 = H_2$, and $f \mapsto \tilde{f} = f$ the identity map, we get

$$\mathbf{Tr}\left(K_{A,2}^{2(n+1)}\right) \leq 2\left[\# A\, e^{-n/B} + \mathbf{Tr}\left(K_{A,1}^{2\lfloor n/(2B)\rfloor}\right)\right] . \tag{2.5}$$

For each $\epsilon \in (0, 1)$, let $U(\epsilon)$ be a finite set containing the identity and such that

$$\mu_2(U(\epsilon)) \geq 1 - \epsilon .$$

As G is amenable, Følner's criterium [16] yields a sequence of increasing finite sets $F(i)$ such that

$$\frac{\# F(i)W}{\# F(i)} \to 1 \tag{2.6}$$

for any fixed finite set W. Fix n and set

$$W = W(n, \epsilon) = U(\epsilon)^{2(n+1)} .$$

Let $A(i) = F(i)W$ and consider the sub-Markovian kernels

$$K_{A(i),j}(x, y), \quad j = 1, 2, \quad i = 1, 2, 3, \dots ,$$

as above. For $j = 1, 2$, let $\mathbf{P}_j^z$ be the probability measure on $G^{\mathbb{N}}$ corresponding to the random walk $(X_\ell)_0^\infty$ on G associated to μ_j (i.e., whose increments $X_\ell^{-1}X_{\ell+1}$ are independent of law μ_j) and started at $X_0 = z$. Hence,

$$\mathbf{P}_j^z(X_n = y) = \mu_j^{(n)}\left(z^{-1}y\right) .$$

In these terms, if $x, z \in A(i)$, we have

$$K_{A(i),j}^n(x, z) = \mathbf{P}_j^x(X_n = z \text{ and } X_\ell \in A(i) \text{ for all } 1 \leq \ell \leq n-1) .$$

It follows that

$$K_{A(i),j}^n(x, x) \leq \mu_j^{(n)}(\mathrm{id}) .$$

Furthermore, if $x \in F(i) \subset A(i) = F(i)W$,

$$\begin{aligned}
&K_{A(i),2}^{2(n+1)}(x, x)\\
&\quad = \mathbf{P}_2^x\left(X_{2(n+1)} = x \text{ and } X_\ell \in A(i) \text{ for all } 1 \leq \ell \leq 2(n+1)\right)\\
&\quad = \mathbf{P}_2^x\left(X_{2(n+1)} = x\right) - \mathbf{P}_2^x\left(X_{2(n+1)} = x \text{ and } \exists\, \ell \in \{1, \dots, 2(n+1)\} : X_\ell \notin A(i)\right)\\
&\quad \geq \mathbf{P}_2^x\left(X_{2(n+1)} = x\right) - \mathbf{P}_2^x\left(\exists\, \ell \in \{0, \dots, 2n+1\} : X_\ell^{-1}X_{\ell+1} \in G \setminus U(\epsilon)\right)\\
&\quad \geq \mu_2^{(2n)}(\mathrm{id}) - 2(n+1)\epsilon .
\end{aligned}$$

The first inequality follows from the inclusion

$$\left\{\forall\ \ell \in \{0, \dots, 2n+1\} : X_\ell^{-1}X_{\ell+1} \in U(\epsilon)\right\} \subset \{\forall\ \ell \in \{0, \dots, 2(n+1)\} : X_\ell \in F(i)W\}$$

which is satisfied when $X_0 \in F(i)$ because $W = U(\epsilon)^{2(n+1)}$. By (2.5) it follows that

$$\begin{aligned}\mu_2^{(2(n+1))}(\mathrm{id}) - 2(n+1)\epsilon &\le \frac{1}{\#F(i)}\mathrm{Tr}\left(K^{2(n+1)}_{A(i),2}\right)\\ &\le 2\left[\frac{\#A(i)}{\#F(i)}e^{-n/B} + \frac{\#A(i)}{\#F(i)}\mu_1^{(2[n/2B])}(\mathrm{id})\right].\end{aligned}$$

Letting i tend to infinity, we obtain

$$\mu_2^{(2(n+1))}(\mathrm{id}) - 2(n+1)\epsilon \le 2\left[e^{-n/B} + \mu_1^{(2[n/2B])}(\mathrm{id})\right].$$

Letting ϵ tend to zero yields the desired inequality. □

Lemma 2.1 and Theorem 2.3 imply Theorem 1.4 since, in this theorem, there is no loss of generality in assuming that G is amenable. Indeed, if G is not amenable and S is a symmetric generating set in G, then $\phi_S(n) \simeq \exp(-n)$ by a celebrated theorem of Kesten [25] whereas, for any symmetric probability measure μ on a countable group, obviously $\phi_\mu(n) \succeq \exp(-n)$.

3. A second proof of $\phi_S \simeq \phi_T$

3.1. Trace comparison in von Neumann algebra

Kenneth Brown pointed out that well-known results in the theory of von Neumann algebra can be used to provide a slicker proof of Theorem 1.1, allowing us to make no distinction between amenable and non-amenable groups. Roughly, the idea is to use a version of the minimax principle belonging to the theory of von Neumann algebra. The usual minimax principle can be interpreted as a comparison of the dimension of certain finite dimensional spectral subspaces. In the von Neumann algebra version, one compares the trace of spectral projections (i.e., the von Neumann dimension of the associated modules). This is well known, but it seems difficult to find a precise reference in text books. Lemma 3 and 4 in [5] give a detailed treatment that suffices for our purpose. To state a precise result, we need some notation. One works in the von Neumann algebra A generated by the reduced C^* algebra of the group G. A symmetric probability measure μ can be interpreted as a bounded self-adjoint operator μ^* acting on $\ell^2(G)$ by $f \to f * \mu$. It belongs to A. If F is a bounded real function of the real variable and $a \in A$ is a self-adjoint bounded operator acting on $\ell^2(G)$, $F(a)$ is a bounded operator on $\ell^2(G)$ defined by elementary functional calculus and $F(a)$ belongs to A. This operator has a bounded convolution kernel that we also denote by $F(a)$ and its trace is defined by

$$\tau[F(a)] = F(a)(\mathrm{id}).$$

For instance, if $F(\lambda) = e^{-t\lambda}$, $t > 0$, and $a = (I - \mu^*)$ for some symmetric probability measure μ on G, then

$$F(a) = e^{-t(I-\mu^*)}$$

has kernel

$$F(a)(y) = e^{-t(\delta-\mu)}(y) = e^{-t}\sum_0^\infty \frac{t^n\mu^{(n)}(y)}{n!}, \quad y \in G$$

where δ is the Dirac mass at $\mathrm{id} \in G$. This is a continuous time convolution semigroup. The next subsection relates the behavior of this continuous time semigroup to the discrete time semigroup $\mu^{(n)}$ in a more general context.

The following lemma follows easily from the arguments in [5, p. 6–7].

Lemma 3.1. *Let G be a finitely generated group (amenable or not). Let μ_1, μ_2 be two symmetric probability measures on G whose Dirichlet forms satisfy $\mathcal{E}_1 \le B\mathcal{E}_2$ (i.e., $(I - \mu_1^*) \le B(I - \mu_2^*)$ in operator notation). Then, for any non-increasing continuous function $F : [0, \infty) \to \mathbb{R}$,*

$$\tau\left[F\left(B\left(I - \mu_2^*\right)\right)\right] \le \tau\left[F\left(\left(I - \mu_1^*\right)\right)\right] .$$

In particular

$$e^{-tB(\delta-\mu_2)}(id) \le e^{-t(\delta-\mu_1)}(id) .$$

3.2. Continuous vs. discrete time

Let (K, π) denote a reversible Markov chain on countable state space X. Consider the iterates K^n and the associated continuous time semigroup

$$H_t = e^{-t(I-K)} = e^{-t}\sum_0^{\infty} \frac{t^n K^n}{n!} .$$

These are self-adjoint operators on $L^2(X, \pi)$. Let $K = \int_{-1}^{+1} \lambda dE_\lambda$ be a spectral resolution of the self-adjoint operator K. Fix $x \in X$, set $f_x = \frac{1_x}{\sqrt{\pi(x)}}$ and consider the probability measure

$$d\mu_x(\lambda) = \langle dE_\lambda f_x, f_x\rangle$$

on $[-1, 1]$. Then

$$K^n(x, x) = \int_{-1}^{1} \lambda^n d\mu_x(\lambda), \quad H_t(x, x) = \int_{-1}^{1} e^{-t(1-\lambda)} d\mu_x(\lambda) .$$

Since $K^{2n+1}(x, x) \ge 0$, it follows that

$$-\int_{-1}^{0} \lambda^{2n+1} d\mu_x(\lambda) \le \int_0^1 \lambda^{2n+1} d\mu_x(\lambda) \le \int_0^1 e^{-(2n+1)(1-\lambda)} d\mu_x(\lambda) \le H_{2n}(x, x)$$

whence

$$K^{2n+2}(x, x) = \int_{-1}^{1} \lambda^{2n+2} d\mu_x(\lambda) \le -\int_{-1}^{0} \lambda^{2n+1} d\mu_x(\lambda) + \int_0^1 \lambda^{2n} d\mu_x(\lambda) \le 2H_{2n}(x, x) .$$

In the other direction, we always have

$$H_{4n}(x, x) = \int_{-1}^{1} e^{-4n(1-\lambda)} d\mu_x(\lambda) \le e^{-2n} + \int_{1/2}^{1} \lambda^{2n} d\mu_x(\lambda) \le e^{-2n} + K^{2n}(x, x) .$$

Thus we can state the following general result.

Proposition 3.2. *Let (K, π) denote a reversible Markov chain on a countable state space X. Then K^n and $H_t = e^{-t(I-K)}$ satisfy*

$$K^{2n+2}(x, x) \le 2H_{2n}(x, x) \quad \text{and} \quad H_{4n}(x, x) \le e^{-2n} + K^{2n}(x, x) .$$

Lemma 2.1, Lemma 3.1, and Proposition 3.2 provide the announced second proof of Theorem 1.4.

4. Invariance under quasi-isometry

The aim of this section is to prove Theorem 1.2. The proof will be adapted from the special case of two different Cayley graphs of a same group treated in Section 2. The approach outlined in Section 3 seems to break down in this setting because quasi-isometries need not preserve invariance at all.

As promised, we now recall the notion of quasi-isometry. Let (X_1, d_1), (X_2, d_2) two metric spaces. We say that a map $\psi : X_1 \to X_2$ is a quasi-isometry if there is a constant C such that

$$\forall\, x, y \in X_1,\quad C^{-1}d_1(x, y) - C \le d_2(\psi(x), \psi(y)) \le Cd_1(x, y) + C\ ; \tag{1}$$

and

$$X_2 = \{x \in X_2 : \exists\, z \in X_1, z \in \psi\,(X_1) \text{ and } d_2(x, z) \le C\}\ . \tag{2}$$

Property (1) says that the map ψ preserves large distances up to multiplicative constants. Property (2) says that ψ is roughly onto. We say that (X_1, d_1), (X_2, d_2) are quasi-isometric if there exists a quasi-isometry from X_1 to X_2. It is not hard to see that this defines an equivalence relation among metric spaces. See [19] and also [9, p. 191] (quasi-isometries are also called rough isometries).

Theorem 1.2 is a corollary of the following result. The method of proof will be similar to that used in Section 2.

Theorem 4.1. *Let (G, S) and (H, T) be the Cayley graphs of two finitely generated groups G, H with S, T finite symmetric generating sets. Denote by d_S and d_T the corresponding graph distance functions on G and H. Assume that there is a map $\psi : G \to H$ and a constant $0 < C < +\infty$ such that*

$$C^{-1}d_S(x, y) - C \le d_T(\psi(x), \psi(y))$$

and

$$H = \{x \in H : \exists\, y \in G, y \in \psi(G) \text{ and } d_T(x, y) \le C\}\ .$$

Then

$$\phi_G(n) \preceq \phi_H(n)$$

where $\phi_G(n) = \mu_S^{(2n)}(id)$, $\mu_S = \frac{1}{\#S}\mathbf{1}_S$ (that is, $\phi_G = \phi_S$) and similarly for ϕ_H.

Before embarking on the proof, let us make a few remarks. Without loss of generality we can assume that G is amenable because, if not, $\phi_G(n) \simeq \exp(-n)$ and $\phi_H(n) \succeq \exp(-n)$. It is also clear that we can replace the given generating set S (or T) by any other finite symmetric generating set as we please. Indeed, such a change will not alter the hypothesis nor the conclusion (by Theorem 1.1).

Proof. We need to introduce some (unfortunately rather cumbersome) notation. By hypothesis, for each $h \in H$, there exists a $\bar{h} \in \psi(G) \subset H$ such that $d_T(h, \bar{h}) \le C$. Fix such a map $h \to \bar{h}$ once and for all with the property that $\bar{h} = h$ if $h \in \psi(G)$. For each h, consider the set $W(h) = \psi^{-1}(\{\bar{h}\}) \subset G$. Since $\bar{h} = h$ when $h \in \psi(G)$, obviously

$$G = \bigcup_{h\in H} W(h)\ . \tag{4.1}$$

By hypothesis, for any fixed h, two elements $u, v \in W(h)$ are at a distance at most C^2 of each other in (G, S). By invariance, this shows that

$$N = \sup_{h\in H} \#W(h)$$

is a finite integer. For each $h \in H$, we use the set $W(h)$ to construct an ordered N-tuple

$$\mathbf{W}(h) = (x_1(h), \ldots, x_N(h)) \tag{4.2}$$

with entries $x_i(h) \in W(h)$ and such that each element of $W(h)$ appears at least once in $\mathbf{W}(h)$.

On $\ell^2(H)$, we have the usual Dirichlet form associated with T which will be denoted $\mathcal{E}_H$ here. It is given by

$$\mathcal{E}_H(f, f) = \frac{1}{2\#T} \sum_{h \in H} \sum_{t \in T} |f(h) - f(ht)|^2$$

for any finitely supported function f. Consider the disjoint union

$$H_N = \{(h, i) : h \in H,\ i \in \{1, \ldots, N\}\}$$

of N copies of H. One of the crucial tools for the proof of Theorem 4.1 will be the one to one linear map

$$f \mapsto \tilde{f} : \ell^2(G) \to \ell^2(H_N)$$

defined by

$$\tilde{f}((h, i)) = f(x_i(h)) \ . \tag{4.3}$$

Not only is this map clearly one to one because of (4.1), but furthermore

$$\|f\|_2 \leq \left\|\tilde{f}\right\|_2 \ . \tag{4.4}$$

On $\ell^2(H_N)$ we define a form $\mathcal{E}_N$ by setting

$$\mathcal{E}_N(f, f) = \frac{1}{2\#T} \sum_{i=1}^{N} \sum_{h \in H} \sum_{t \in T} |f((h, i)) - f((ht, i))|^2 \ . \tag{4.5}$$

Observe that this is the Dirichlet form associated with simple random walk on the disconnected graph obtained as the disjoint union of N copies of (H, T). That is, if we set

$$K_N((x, i), (y, j)) = \begin{cases} \mu_T\left(x^{-1}y\right) & \text{if } i = j \\ 0 & \text{otherwise}, \end{cases} \tag{4.6}$$

then $\mathcal{E}_N$ can be written

$$\mathcal{E}_N(f, f) = \langle (I - K_N) f, f \rangle$$

where the scalar product is in $\ell^2(H_N)$ and

$$K_N f((x, i)) = \sum_{(y,j) \in H_N} K_N((x, i), (y, j)) f((y, j)) \ .$$

Now, we are going to pick a convenient finite symmetric generating set in G. We will call it R. If h, h' are two elements that are neighbors in (H, T), i.e., such that $h' = ht$ for some $t \in T$, then $\bar{h}, \bar{h}'$ are at distance at most $2C + 1$ of each other. Hence, any $g \in W(h)$, $g' \in W(h')$ are at distance at most $(3C + 1)C$ of each other in (G, S). We set

$$R = \{z \in G : d_S(\mathrm{id}_G, z) \leq (3C + 1)C\} \ .$$

This is, of course, a finite symmetric generating set in G and we set

$$\mathcal{E}_G(f,f) = \frac{1}{2\#R} \sum_{g\in G, r\in R} |f(g) - f(gr)|^2 .$$

This construction makes the following lemma essentially obvious.

Lemma 4.2. *Referring to notation introduced above, there exists a constant B such that*

$$\mathcal{E}_N\left(\tilde{f},\tilde{f}\right) \le B\mathcal{E}_G(f,f)$$

for all finitely supported functions in $\ell^2(G)$.

Proof. Indeed,

$$\begin{aligned}
\mathcal{E}_N\left(\tilde{f},\tilde{f}\right) &= \frac{1}{2\#T}\sum_{i=1}^{N}\sum_{h\in H}\sum_{t\in T}\left|\tilde{f}((h,i)) - \tilde{f}((ht,i))\right|^2 \\
&= \frac{1}{2\#T}\sum_{i=1}^{N}\sum_{h\in H}\sum_{t\in T}|f(x_i(h)) - f(x_i(ht))|^2 .
\end{aligned}$$

By definition of the generating set $R \subset G$, if $t \in T$ then $x_i(ht) = x_i(h)r$ for some $r \in R$. Hence

$$\begin{aligned}
\mathcal{E}_N\left(\tilde{f},\tilde{f}\right) &\le \frac{1}{2\#T}\sum_{i=1}^{N}\sum_{h\in H}\sum_{t\in T}\sum_{r\in R}|f(x_i(h)) - f(x_i(h)r)|^2 \\
&\le \frac{NM}{2}\sum_{g\in G}\sum_{r\in R}|f(g) - f(gr)|^2 \\
&\le \#R\, NM\mathcal{E}_G(f,f)
\end{aligned}$$

where

$$M = \sup_{h\in H}\#\left\{h' \in H : \bar{h}' = \bar{h}\right\} .$$

By invariance, it is clear that M is finite since two elements h, h' with $\bar{h} = \bar{h}'$ are at a distance at most $2C$ of each other. This ends the proof of Lemma 4.2. □

Fix a finite set $A \subset G$. Consider the sub-Markov kernel $K_A = K_{A,\mu_R}$ defined at (2.2) and the associate Dirichlet form given by (2.3).

Associate to $A \subset G$ the finite subset A_N of H_N given by

$$A_N = \{(h,i) \in H_N : x_i(h) \in A\} .$$

On A_N there is a natural sub-Markovian kernel defined by

$$K_{N,A}\left((h,i),(h',j)\right) = \begin{cases} \mu_T\left(h^{-1}h'\right) & \text{if } (h,i),\left(h',j\right) \in A_N \text{ and } i = j \\ 0 & \text{otherwise} \end{cases}$$

with associated Dirichlet form

$$\mathcal{E}_{N,A}(f,f) = \left\langle \left(I - K_{N,A}\right) f, f\right\rangle$$

where the scalar product is in $\ell^2(A_N)$.

For f defined on A (resp. A_N), we define the extension $\hat{f}$ of f to G (resp. H_N) by setting $\hat{f} = f$ on A (resp. A_N) and $\hat{f} = 0$ otherwise. Then

$$\mathcal{E}_A(f, f) = \mathcal{E}_G\left(\hat{f}, \hat{f}\right) \quad \left(\text{resp. } \mathcal{E}_{N,A}(f, f) = \mathcal{E}_N\left(\hat{f}, \hat{f}\right)\right) . \tag{4.7}$$

The map $f \mapsto \tilde{f}$ defined by (4.3) induces by restriction a map from $\ell^2(A)$ to $\ell^2(A_N)$ and it is obvious from the definition that the operations $\hat{}$ and $\tilde{}$ commute: For any $f \in \ell^2(A)$ and all $(h, i) \in H_N$, we have

$$\left(\hat{f}\right)^{\sim}((h,i)) = \left(\tilde{f}\right)^{\wedge}((h,i)) . \tag{4.8}$$

Hence (4.4), (4.7), (4.8), and Lemma 4.2 show that the map

$$f \mapsto \tilde{f} : \ell^2(A) \to \ell^2(A_N)$$

is a linear one to one map satisfying

$$\|f\|_2 \leq \left\|\tilde{f}\right\|_2 \quad \text{and} \quad \mathcal{E}_{N,A}\left(\tilde{f}, \tilde{f}\right) \leq B\mathcal{E}_A(f, f) \tag{4.9}$$

with B the constant from Lemma 4.2. Lemma 2.2 and (4.9) yield

$$\mathbf{Tr}\left(K_A^{2n+2}\right) \leq 2\left[N\#Ae^{-n/B} + \mathbf{Tr}\left(K_{N,A}^{2[n/(2B)]}\right)\right] . \tag{4.10}$$

As noticed before, we can assume that G is amenable. Thus, let $F(i) \subset G$, $i = 1, 2 \ldots,$ a Følner sequence in G as in (2.6)) so that $\lim_{i\to\infty} \frac{\#F(i)U}{\#F(i)} = 1$ for any finite set $U \in G$. Fix an integer $n \geq 1$, set $U = R^{4n}$ and $A(i) = F(i)U$. After $2n + 2$ steps the random walk in G of law μ_R started at $x \in F(i)$ cannot ever have exit $A(i)$. Hence, for $x \in F(i)$,

$$\mathbf{P}_R^x(X_{2n+2} = x) = \phi_R(n+1) = K_{A(i)}^{2n+2}(x, x) .$$

Furthermore, it is clear that

$$K_{N,A}^n((h,i),(h,i)) \leq \mathbf{P}_T^h\,(X_n = h) = \mu_T^{(n)}\,(\mathrm{id}_H) .$$

Hence, (4.10) yields

$$\phi_R(n+1) \leq \frac{1}{\#F(i)}\mathbf{Tr}\left(K_A^{2n+2}\right) \leq \frac{2N\#A(i)}{\#F(i)}\left(e^{-n/B} + \phi_T(2[n/(2B)])\right) .$$

Letting i tend to infinity, we obtain

$$\phi_R(n+1) \leq 2N\left(e^{-n/B} + \phi_T(2[n/(2B)])\right)$$

which is the desired inequality. This ends the proof of Theorem 4.1. Theorem 1.2 follows. □

5. Heat diffusion kernel on covering manifolds

Let M be a Riemannian manifold, G be a finitely generated group of isometries of M such that $M/G = N$ is a compact Riemannian manifold. Thus M is a covering of N with deck transformation group G. Let Δ denote the Laplace-Beltrami operator on M and let $h_t(x, y)$ be the smooth positive kernel of the heat diffusion semigroup $e^{-t\Delta}$. Set

$$\Phi_M(t) = \sup_{x \in M} \min\{1, h_t(x, x)\} .$$

Observe that, by invariance under G and compactness of N, one can prove that

$$\sup_{x \in M} h_t(x, x) \simeq h_t(y, y)$$

as functions of t, for any fixed $y \in M$. Also note that the only purpose of taking $\min\{1, h_t(x, x)\}$ in the definition of $\Phi_M(t)$ is to eliminate the behavior of h_t for small t which of course is irrelevant for comparison with the behavior of random walk on G.

Let S be a finite symmetric generating set in G and let

$$\phi_G(n) = \phi_S(n) = \mathbf{P}_S^{\mathrm{id}}(X_{2n} = \mathrm{id})$$

as before. The aim of this section is to prove the following theorem.

Theorem 5.1. *Let M be a covering of a compact manifold N with deck transformation group G. Then, referring to the notation introduced above,*

$$\Phi_M \simeq \phi_G .$$

This theorem says that up the equivalence relation $\simeq$, the behavior of the heat diffusion on M (for large t) is the same as the behavior of the random walk on G. By a Theorem of Brooks [4] (see also [30]), we know that G is non-amenable if and only if the spectrum of Δ is contained in $[\lambda_0, +\infty)$ with $\lambda_0 > 0$. Moreover, in this case, both Φ_M and ϕ_G decay exponentially fast. Thus, it is enough to prove the theorem when G is amenable.

5.1. Notation and background

Let M be a covering of a compact manifold N with deck transformation group G as above. Let $|\nabla f|$ denote the length of the gradient of a function f. The Dirichlet form associated to the Laplace-Beltrami operator Δ on M is given by

$$\|\nabla f\|_2^2 = \langle \Delta f, f\rangle$$

for any smooth compactly supported function f. Here, the scalar product is in $L^2(M, dv)$, dv being the Riemannian volume.

Denote the Riemannian distance function on M by

$$M \times M \ni (x, y) \mapsto d(x, y)$$

and set

$$B(x, r) = \{y \in M : d(x, y) \le r\}, \quad V(x, r) = \mathrm{Vol}\,(B(x, r)) .$$

For any subset $W \subset M$ and $r > 0$, define

$$W^+(r) = \{x \in M : d(x, W) < r\} \quad \text{and} \quad W^-(r) = \left\{x \in W : d\left(x, W^c\right) > r\right\}$$

so that, obviously,

$$W^-(r) \subset W \subset W^+(r) .$$

Since M is a covering of a compact manifold, for any fixed $0 < r \leq R < +\infty$, there exist two constants $C_0(R)$, $C_0(r, R)$ such that

$$\max_{0<t\leq R}\left\{\frac{\max_{x\in M} V(x,4t)}{\min_{x\in M} V(x,t)}\right\} \leq C_0(R) \tag{5.1}$$

and

$$\frac{\max_{x\in M} V(x,R)}{\min_{x\in M} V(x,r)} \leq C_0(r,R)\,. \tag{5.2}$$

Fix $o \in M$, a base point and identify G to a subset of M through $G \ni g \to go$. The group G, viewed as a subset of M, has the following two properties:

$$\begin{cases} (1) & \exists\, r_0 > 0 : g \neq g' \Rightarrow d\left(g, g'\right) \geq r_0 \\ (2) & \exists\, R_0 < \infty : \max_{x\in M} d(G,x) < R_0\,. \end{cases} \tag{5.3}$$

Of course, we can assume that $r_0 \leq R_0$. Set

$$\Omega = \{x \in M : \forall\, g \in G,\ g \neq \mathrm{id}, d(o,x) < d(g,x)\}\,. \tag{5.4}$$

This set is a fundamental domain for the action of G on M, that is,

$$g\Omega \cap g'\Omega = \emptyset \quad \text{if } g \neq g' \quad \text{and} \quad M = \bigcup_{g\in G} g\overline{\Omega}\,.$$

Observe that

$$B\left(o, r_0/2\right) \subset \overline{\Omega} \subset B\left(o, R_0\right)\,.$$

Let $W \subset M$ be an open set. We will need to consider the sub-Markovian heat semigroup H_t^W acting on $L^2(W, dv)$ associated to the Laplacian Δ with Dirichlet boundary condition in W. Technically, this semigroup has generator $-\Delta^W$ where Δ^W is the Friedrichs extension of the symmetric operator Δ with domain $C_0^\infty(W)$, the space of smooth functions with compact support in W. See, e.g., [15, p. 11]. Let $h_t^W(x,y)$, $x, y \in W, t > 0$, denote the kernel of H_t^W. This kernel describes the probability of going from x to y in time t without leaving W. When W is relatively compact in M it is well known that the spectrum of Δ_W is made of a sequence of positive eigenvalues $\lambda_W(i) \nearrow +\infty$, $i = 1, 2, \ldots$, (repeated according to their multiplicity and in non-decreasing order).

Later on we will need a number of known results concerning h_t^W that we now recall. Consider the diffusion (i.e., Brownian motion) $(X_t)_{t>0}$ on the Riemannian manifold M and let $\tau = \tau_W$ be the stopping time

$$\tau = \inf\{t : t \geq 0, X_t \notin W\}\,.$$

Thus τ is the first exit time from W and X_τ is the position of the process when it exits W. The strong Markov property yields the well-known formula (see [23])

$$\forall\, x, y \in W, \quad h_t^W(x,y) = h_t(x,y) - E^x\left(h_{t-\tau}(X_\tau, y)\mathbf{1}_{\{\tau\leq t\}}\right)$$

where $E^x(\cdot)$ denote the expectation with respect to $(X_t)_{t>0}$ started at $X_0 = x$. Hence,

$$h_t^W(x,y) \leq h_t(x,y)$$

for any open set W. Furthermore,

$$h_t^W(x,y) \geq h_t(x,y) - \sup_{0<s<t}\ \sup_{z\in\partial W}\{h_s(z,y)\}\,.$$

In the present situation the heat kernel satisfies

$$h_t(z,y) \le C_1(\min\{1,t\})^{-d/2}\exp\left(-d(z,y)^2/C_1t\right), \tag{5.5}$$

where d is the dimension of M. See, for instance, [29, Theorem 6.1]. The earliest reference for such an estimate seems to be [10].

Lemma 5.2. *There exists $D > 0$ and for each $\epsilon > 0$ there exists $\alpha > 0$ such that for any open set W and any $t \ge 1$, we have*

$$\forall\, x \in W^-\left(\alpha t^{1/2}\right), \quad h_t^W(x,x) \ge D^{-1}\max_{y\in M}\left\{h_{t/2}(y,y)\right\} - \epsilon$$

where $W^-(r) = \{x \in W : d(x,\partial W) > r\}$.

Proof. For $t \ge 1$ and $y \in W^-(\alpha t^{1/2})$, (5.5) yields

$$\sup_{0<s<t}\sup_{z\in\partial W}\{h_s(z,y)\} \le C_1\sup_{0<s<t}(\min\{1,s\})^{-d/2}\exp\left(-\alpha^2 t/C_1 s\right) \le C_1'\alpha^{-d}.$$

Hence, if α is such that $C_1'\alpha^{-d} = \epsilon$,

$$\forall\, x \in W^-\left(\alpha t^{1/2}\right), \quad h_t^W(x,x) \ge h_t(x,x) - \epsilon.$$

But, by the Li-Yau parabolic Harnack inequality (see, e.g., [15, Theorem 5.3.5]) and the fact that M/G is compact,

$$\forall\, x \in M, \ \forall\, t \ge 1, \quad \max_{y\in M} h_{t/2}(y,y) \le Dh_t(x,x)$$

where D depends only on M. The desired result follows. □

Lemma 5.3. *There exists a constant C_2 depending only on M such that, for any open bounded $W \subset M$,*

$$\int_W h_t^W(x,x)dv(x) = \sum_{i=1}^{\infty} e^{-t\lambda_W(i)} \le \sum_{i:\lambda_W(i)\le 1} e^{-t\lambda_W(i)} + C_2\mathrm{Vol}\,(W)\,e^{-t/2}.$$

Proof. The heat kernel estimate (5.5) shows that

$$h_{1/2}^W(x,x) \le h_{1/2}(x,x) \le C_2$$

where $C_2 = C_1 2^{d/2}$ is independent of $x \in M$ and W. Hence, for $t \ge 1$,

$$\begin{aligned}
\int_W h_t^W(x,x)dv(x) &\le \sum_{i:\lambda_W(i)\le 1} e^{-t\lambda_W(i)} + e^{-(t-1/2)}\sum_{i:\lambda_W(i)>1} e^{-\lambda_W(i)/2} \\
&\le \sum_{i:\lambda_W(i)\le 1} e^{-t\lambda_W(i)} + e^{-t/2}\sum_{1}^{\infty} e^{-\lambda_W(i)/2} \\
&\le \sum_{i:\lambda_W(i)\le 1} e^{-t\lambda_W(i)} + e^{-t/2}\int_W h_{1/2}^W(x,x)dv(x) \\
&\le \sum_{i:\lambda_W(i)\le 1} e^{-t\lambda_W(i)} + C_2\,\mathrm{Vol}\,(W)\,e^{-t/2}.
\end{aligned}$$

□

5.2. Proof of $\phi_G \preceq \Phi_M$

Let r_0, R_0 be as defined at (5.3). It will turn out to be convenient to choose the finite generating set S of G so that

$$\{g : d(o, g) \leq 4R_0\} \subset S . \tag{5.6}$$

We can now construct a partition of unity on M indexed by G, say θ_g, $g \in G$, so that

$$\forall\, x \in M, \quad \sum_g \theta_g(x) = 1, \quad \theta_g \geq 0$$

and

$$\theta_g \equiv 0 \text{ outside } B(g, 2R_0), \quad \theta_g \equiv 1 \text{ in } B(g, r_0/4), \quad |\nabla\theta_g| \leq C .$$

Let us recall briefly this construction. Let Ω be the fundamental domain defined at (5.4). Let $\eta \geq 0$ be a smooth bump function around $o \in M$ with the properties that support(η) $\subset \Omega^+(r_0/4)$ and $\eta = 1$ on Ω. Observe that this implies that

$$\eta\left(g^{-1}x\right) = 0 \text{ if } g \neq \text{id and } x \in B(o, r_0/4)$$

and set

$$\theta_g(x) = \frac{\eta\left(g^{-1}x\right)}{\sum_{h\in G}\eta\left(h^{-1}x\right)} .$$

The desired properties easily follow from this definition after observing that

$$\min_{x\in M} \sum_{h\in G} \eta\left(h^{-1}x\right) \geq 1$$

and

$$\max_{x\in M} \#\left\{h : \eta\left(h^{-1}x\right) > 0\right\} < +\infty .$$

The first assertion is obvious since, for each x, there is at least one h such that $h^{-1}x \in \overline{\Omega}$ and $\eta = 1$ on $\overline{\Omega}$. The second assertion can be proved as follows. Fix $x \in M$. If h is such that $\eta(h^{-1}x) > 0$, then $d(x, h) \leq 2R_0$. For each such h, consider the open ball of radius $r_0/2$ centred at $h \in M$. These balls are pairwise disjoint. Hence, by (5.2),

$$\max_{x\in M} \#\left\{h : \eta\left(h^{-1}x\right) > 0\right\} \leq \frac{\max_x V(x, 2R_0)}{\min_x V(x, r_0/2)} \leq C_0(r_0/2, 2R_0) .$$

Observe that, for each g and all $x \in B(g, r_0/4)$,

$$\theta_g(x) = 1 \text{ whereas } \theta_h(x) = 0 \text{ if } h \neq g . \tag{5.7}$$

We consider the linear map $f \mapsto \tilde{f} : \ell^2(G) \to L^2(M)$ defined by

$$\tilde{f}(x) = \sum_{g\in G} f(g)\theta_g(x) .$$

This map has the following two properties:

$$\forall\, f \in \ell^2(G), \quad \|f\|_2 \leq C_3 \left\|\tilde{f}\right\|_2 \tag{5.8}$$

$$\forall\, f \in \ell^2(G), \quad \left\|\nabla\tilde{f}\right\|_2 \leq C_3\mathcal{E}_S(f, f) . \tag{5.9}$$

Inequality (5.8) is clear because, by (5.7), there is a ball of radius $r_0/4$ around each $g \in M$ such that $\tilde{f}(x) = f(g)$ on that ball. To prove (5.9) observe that $\nabla \sum_g \theta_g \equiv 0$. Hence, if $x \in B(g, R_0)$,

$$\nabla \tilde{f}(x) = \sum_h (f(h) - f(g)) \nabla \theta_h(x) .$$

If $\theta_h(x) \neq 0$, then $d(x, h) \leq 2R_0$ so that $d(h, g) \leq 3R_0$. By (5.6), it follows that

$$\int_{B(g,R_0)} \left|\nabla \tilde{f}(x)\right|^2 dv(x) \leq C^2 \sum_{s \in S} |f(gs) - f(g)|^2$$

since $|\nabla \theta_h| \leq C$ for all $h \in G$. This implies the desired conclusion with $C_3 = 2C^2 \# S$ because the balls $B(g, R_0)$, $g \in G$, cover M.

Now, fix a finite set $A \subset G$ and set

$$A_0 = \{x \in M : d(x, A) < 2R_0\} .$$

This is a bounded subset of M. On $\ell^2(A)$, consider the sub-Markovian operator K_A with Dirichlet form $\mathcal{E}_A$ associated to μ_S as in (2.2). On A_0 consider the heat diffusion semigroup with Dirichlet boundary condition $H_t^{A_0}$.

Any $f \in \ell^2(A)$ can be extended trivially outside A by setting $f(g) = 0$ if $g \notin A$ and this allows us to define $\tilde{f}$ for $f \in \ell^2(A)$. Observe that if $f \in \ell^2(A)$, then $\tilde{f}$ is a smooth function with compact support in A_0 and that (5.8) and (5.9) hold true with $\mathcal{E}_S(f, f) = \mathcal{E}_A(f, f)$. See (2.3). Also, observe that

$$\mathrm{Vol}\,(A_0) \leq C_4 \# A \tag{5.10}$$

for some constant C_4 independent of A.

Let $\beta_A(i)$, $1 \leq i \leq \#A$ be the eigenvalues of K_A on $\ell^2(A)$ in non-increasing order. Let $\lambda_{A_0}(i)$, $i = 1, \ldots,$ be the Dirichlet eigenvalues of Δ in A_0, in non-decreasing order. Then, (5.8), (5.9), and the minimax principle yield

$$\lambda_{A_0}(i) \leq B\,(1 - \beta_A(i)) \quad i = 1, \ldots, \#A ,$$

with $B = C_3^2$, C_3 from (5.8) and (5.9). Taking traces, and using the argument of Lemma 2.2 to take care of those $\beta_A(i)$'s that are negative, we get

$$\mathbf{Tr}\left(K_A^{2(n+1)}\right) \leq 2\,\mathbf{Tr}\left(H_{2n/B}^{A_0}\right) \tag{5.11}$$

where the left-hand side is as in (2.4) and

$$\mathbf{Tr}\left(H_t^{A_0}\right) = \sum_i e^{-t\lambda_{A_0}(i)} = \int_{A_0} h_t^{A_0}(x, x) dv(x) .$$

From here, assuming (as we may) that G is amenable, we choose a Følner sequence $F(i)$ as in (2.6). Fixing n and applying (5.11) and (5.10) to each $A(i) = F(i)S^{4n}$, we obtain (see the end of Section 4 for the first inequality)

$$\begin{aligned}
\phi_S(n+1) &\leq \frac{1}{\#F(i)} \mathbf{Tr}\left(K_A^{2(n+1)}\right) \\
&\leq 2\frac{\mathrm{Vol}(A_0)}{\#F(i)} \sup_{x \in A_0} h_{2n/B}^{A_0}(x, x) \\
&\leq 2C_4 \frac{\#A(i)}{\#F(i)} \sup_{x \in M} h_{2n/B}(x, x) .
\end{aligned}$$

Letting i tend to infinity now yields

$$\phi_S(n) \leq 2C_4 \sup_{x \in M} h_{2n/B}(x, x)$$

since $\#A(i)/\#F(i) \to 1$ by (2.6). The desired conclusion

$$\phi_S(n) \leq 2C_4 \Phi_M(2n/B)$$

follows since $\phi_S(n) \leq 1$ and

$$\Phi_M(t) = \sup_{x \in M} \min\{1, h_t(x, x)\} .$$

5.3. Proof of $\Phi_M \preceq \phi_G$

This is the most technical part of the proof of Theorem 5.1. The additional ingredient we need is a good discrete approximation of M. We will appeal to a Poincaré inequality proved by Buser in [6] for manifolds with Ricci curvature bounded below (our M has Ricci curvature bounded below since M/G is compact). See also [9, p. 288]. Buser's inequality can be stated as follows. For each $R > 0$ there exists a constant $P_0(R)$ such that

$$\int_B |f - f_B|^2 \, dv \leq P_0(R) r^2 \int_B |\nabla f|^2 dv \tag{5.12}$$

for any ball $B = B(x, r) \subset M$ of radius r, $0 < r \leq R$, and any Lipschitz function f in B. Here f_B denotes the mean of f over B.

For each $\delta \in (0, 1)$, let $\Gamma = \Gamma(\delta) \subset \overline{\Omega}$ be a maximal finite set of points of Ω such that

$$d\left(\gamma, \gamma'\right) \geq \delta/2 \quad \text{if} \quad \gamma, \gamma' \in \Gamma(\delta) \ \text{ and } \ \gamma \neq \gamma' .$$

From the maximality of Γ, it follows that

$$\Omega \subset \cup_{\gamma \in \Gamma} B(\gamma, \delta) .$$

Furthermore, the balls $B(\gamma, \delta)$, $\gamma \in \Gamma$ do not overlap to much since, by (5.1),

$$\forall\, x \in M, \quad \#\{\gamma \in \Gamma : x \in B(\gamma, \delta)\} \leq \frac{\max_{y \in M} V(y, 2\delta)}{\min_{y \in M} V(y, \delta/2)} \leq C_0(1) . \tag{5.13}$$

Write

$$\Gamma = \{\gamma_1, \cdots, \gamma_N\} ,$$

$N = N(\delta)$ being the cardinality of Γ. Consider the disjoint union G_N of N copies of G, i.e.,

$$G_N = \{(g, i) : g \in G, i \in \{1, \ldots, N\}\} .$$

Given a function $f : M \to \mathbb{R}$ define $\tilde{f} : G_N \mapsto \mathbb{R}$ by setting

$$\tilde{f}(g, i) = \frac{1}{V(g\gamma_i, \delta)} \int_{B(g\gamma_i, \delta)} f(z) dv(z) . \tag{5.14}$$

Lemma 5.4. *The linear map $f \mapsto \tilde{f}$, restricted to Lipschitz functions with compact support, has the following properties:*

(1) *There exist two constant C_5, C_5' independent of $\delta \in (0,1)$ such that*

$$\|f\|_2^2 \le C_5' \left\|\tilde f\right\|_2^2 + C_5\delta^2\|\nabla f\|_2^2 .$$

(2) *For each $\delta \in (0,1)$ there exists a constant $C(\delta)$ such that*

$$\mathcal{E}_N\left(\tilde f, \tilde f\right) \le C(\delta)\|\nabla f\|_2^2$$

where

$$\mathcal{E}_N(f,f) = \frac{1}{2\#S}\sum_{i=1}^{N}\sum_{g\in G, s\in S} |f((g,i)) - f((gs,i))|^2 .$$

Proof of (1). The balls $B(g\gamma_i,\delta)$, $g \in G$, $\gamma_i \in \Gamma$ cover M. Hence,

$$\begin{aligned}
\|f\|_2^2 &\le \sum_{g\in G}\sum_{i=1}^{N}\int_{B(g\gamma_i,\delta)} |f|^2 dv \\
&\le 2\sum_{g\in G}\sum_{i=1}^{N} V(g\gamma_i,\delta)\left|\tilde f(g,i)\right|^2 + 2\sum_{g\in G}\sum_{i=1}^{N}\int_{B(g\gamma_i,\delta)}\left|f - \tilde f(g,i)\right|^2 dv \\
&\le 2\max_{z\in M} V(z,1)\left\|\tilde f\right\|_2^2 + 2P_0(1)\delta^2\sum_{g\in G}\sum_{i=1}^{N}\int_{B(g\gamma_i,\delta)}|\nabla f|^2 dv \\
&\le 2\max_{z\in M} V(z,1)\left\|\tilde f\right\|_2^2 + 2P_0(1)C_0(1)\,\delta^2\int_M |\nabla f|^2 dv .
\end{aligned}$$

Here we have used the Poincaré inequality (5.12) and (5.13), i.e., the fact that the balls $B(g\gamma_i,\delta)$, $g \in G, i = 1,\dots,N$, do not overlap much. This yields the desired conclusion with $C_5' = 2\max_{z\in M} V(z,1)$ and $C_5 = 2P_0(1)C_0(1)$. □

Proof of (2). Observe that there exists $R_1 > 0$ such that, for all $\delta \in (0,1)$,

$$\bigcup_{s\in S\cup\{\mathrm{id}\}}\bigcup_{i=1}^{N} B(s\gamma_i,\delta) \subset B(o,R_1) .$$

Write

$$\begin{aligned}
\mathcal{E}_N\left(\tilde f,\tilde f\right) &= \frac{1}{2\#S}\sum_{i=1}^{N}\sum_{g\in G,s\in S}\left|\tilde f((g,i)) - \tilde f((gs,i))\right|^2 \\
&= \frac{1}{2\#S}\sum_{i=1}^{N}\sum_{g\in G,s\in S}\left|\frac{1}{V(g\gamma_i,\delta)}\int_{B(g\gamma_i,\delta)} f dv - \frac{1}{V(gs\gamma_i,\delta)}\int_{B(gs\gamma_i,\delta)} f dv\right|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2\#S\min_z V(z,\delta)^2}\sum_{i=1}^{N}\sum_{g\in G,s\in S}\int_{B(g\gamma_i,\delta)}\int_{B(gs\gamma_i,\delta)}|f(x)-f(y)|^2dv(x)dv(y)\\
&\leq \frac{N}{2\min_z V(z,\delta)^2}\sum_{g\in G}\int_{B(g,4R_1)}\int_{B(g,4R_1)}|f(x)-f(y)|^2dv(x)dv(y)\\
&\leq \frac{N\,P_0(4R_1)\max_z V(z,4R_1)}{4\min_z V(z,\delta)^2}\sum_{g\in G}\int_{B(g,4R_1)}|\nabla f|^2dv\\
&\leq C(\delta)\int_M|\nabla f|^2dv\,.
\end{aligned}
$$

The last step uses the fact that the balls $B(g,R_1)$, $g\in G$, do not overlap too much and (5.2). □

Fix a finite subset $A\subset G$ and let

$$A\Omega=\bigcup_{g\in A}g\Omega\,.$$

Let $U(A)\subset M$ be an open set with smooth boundary and the following property

$$U(A)\subset[A\Omega]^-(r_0/4)\subset A\Omega\subset U(A)^+(r_0)\,. \tag{5.15}$$

Since $U(A)$ has a smooth boundary and is bounded, one can use classical elliptic theory (e.g., [20, Lemma 6.4]) to see that any function u solution of

$$\left\{\begin{array}{lcl}\Delta u&=&\lambda u\\ u&=&0\ \text{ on }\partial U\end{array}\right. \tag{5.16}$$

for some $\lambda\geq 0$ has a bounded gradient in U(A). It follows that the function

$$\hat{u}(x)=\left\{\begin{array}{ll}u(x)&\text{if } x\in U(A)\\ 0&\text{otherwise}\end{array}\right.$$

is a Lipschitz function on M. Moreover, if $\delta\in(0,1)$ is small enough (e.g., $\delta\leq r_0/4$) then, by (5.14) and (5.15), the function

$$\check{u}=\widetilde{(\hat{u})}:G_N\mapsto\mathbb{R}$$

has support in

$$A_N=\{(g,i):g\in A,i\in\{1,\dots,N\}\}\,.$$

On A_N, we consider the self-adjoint operator

$$K_{N,A}\left((g,i),(g',j)\right)=\left\{\begin{array}{ll}\mu_S\left(g^{-1}g'\right)&\text{if } (g,i),(g',j)\in A_N \text{ and } i=j\\ 0&\text{otherwise}\end{array}\right.$$

with associated Dirichlet form $\mathcal{E}_{N,A}(f,f)=\langle(I-K_{N,A})f,f\rangle$, $f\in\ell^2(A_N)$.

Lemma 5.5. *Let $\mathcal{H}_A$ be the finite dimensional subspace of $L^2(U(A),dv)$ spanned by the solutions of (5.16) with $0<\lambda\leq 1$. We can choose $\delta\in(0,1)$ small enough so that, for any finite set $A\subset G$, the linear map*

$$u\to\check{u}=\widetilde{(\hat{u})}:\mathcal{H}_A\subset L^2(U(A),dv)\to\ell^2(A_N)$$

satisfies

$$\|u\|_2^2 \le C_6 \left\|\check{u}\right\|_2^2$$

and

$$\mathcal{E}_{N,A}\left(\check{u},\check{u}\right) \le C_6 \|\nabla u\|_2^2 .$$

Proof. The functions in $\mathcal{H}_A$ are Lipschitz with compact support. Thus by Lemma 5.4, for any small $\delta > 0$,

$$\|u\|_2^2 = C_5' \left\|\hat{u}\right\|_2^2 \le \left\|\check{u}\right\|_2^2 + C_5\delta^2 \|\nabla u\|_2^2$$

and

$$\mathcal{E}_{N,A}\left(\check{u},\check{u}\right) = \mathcal{E}_N\left(\check{u},\check{u}\right) \le C(\delta)\|\nabla u\|_2^2 .$$

But $\|\nabla u\|_2^2 = \langle \Delta u, u\rangle \le \|u\|_2^2$ because u is solution of (5.16) with $0 < \lambda \le 1$. The desired result follows if we pick any $\delta \le C_5^{-1/2}/2$. Observe that fixing δ determines the value of $N = N(\delta)$. □

By the minimax principle, Lemma 5.5 yields a comparison between the Dirichlet eigenvalues $0 < \lambda_{U(A)}(i) \le 1$ of Δ in $U(A)$ in non-decreasing order, say there are k of them, and the eigenvalues $\beta_{N,A}(i)$ (in non-increasing order) of $K_{N,A}$ on $\ell^2(A_N)$. Namely,

$$1 - \beta_{N,A}(i) \le B\lambda_{A(U)}(i), \quad i = 1, \dots, k$$

with $B = C_6^2$. Thus, if $\beta_{N,A}(i) \ge 1/2$, then $e^{-2n\lambda_{A(U)}(i)} \le \beta_{N,A}(i)^{n/B}$. Taking traces as in the proof of Lemma 2.2 gives

$$\sum_{i:\lambda_{U(A)}(i)\le 1} e^{-2n\lambda_{U(A)}(i)} \le \mathbf{Tr}\left(K_{N,A}^{2[n/2B]}\right) + N\#A\, e^{-n/B} .$$

From the definition of $K_{N,A}$, it is clear that, for any $j \in \{1, \dots, N\}$,

$$K_{N,A}^n((g,j),(g,j)) \le \mathbf{P}_S^g\left(X_n = g\right) = \mu_S^{(n)}(\mathrm{id}) .$$

Hence,

$$\sum_{i:\lambda_{U(A)}(i)\le 1} e^{-2n\lambda_{U(A)}(i)} \le N\#A\left(\phi_G([n/(2B)]) + e^{-n/B}\right) . \tag{5.17}$$

Fix an integer n. Fix $\epsilon > 0$ and let $\alpha = \alpha(\epsilon)$ be as given by Lemma 5.2. Pick $T = T(n,\epsilon) \subset G$ to be a finite subset of G so large that

$$A\Omega = \bigcup_{g\in A} \Omega \subset U(AT)^-\left((2n)^{1/2}\alpha\right)$$

for all finite set $A \subset G$. For instance

$$T = \left\{g \in G : d(o,g) \le (2n)^{1/2}\alpha + 10R_0\right\}$$

do the job. Then, by Lemma 5.2,

$$\max_{y\in M}\left\{h_{t/2}(y,y)\right\} \le D\epsilon + \frac{D}{\mathrm{Vol}\,(A\Omega)} \int_{A\Omega} h_t^{U(AT)}(x,x)dv(x)$$

for $t \geq 1$. Thus, by Lemma 5.3 and (5.17),

$$\begin{aligned}
&\max_{y \in M} \{h_n(y, y)\} \\
&\leq \; D\epsilon + \frac{D}{\mathrm{Vol}\,(A\Omega)} \left(\sum_{i:\lambda_{U(AT)}(i) \leq 1} e^{-2n\lambda_{U(AT)}(i)} + C_2 \,\mathrm{Vol}\,(U(AT))\, e^{-n} \right) \\
&\leq \; D\epsilon + \frac{D}{\mathrm{Vol}\,(A\Omega)} \left(N\#[AT] \left(\phi_G([n/(2B)]) + e^{-n/B} \right) + C_2 \,\mathrm{Vol}\,(U(AT))\, e^{-n} \right) \\
&\leq \; D\epsilon + C_7 \frac{\#[AT]}{\#A} \left(\phi_G([n/(2B)]) + e^{-n/B} \right) .
\end{aligned}$$

Finally, assuming as we may that G is amenable, let $F(i)$ be a Følner sequence as in (2.6). For each i, apply the last inequality to $A = F(i)$ and let i tend to infinity. Since $\#[F(i)T]/\#F(i) \to 1$, we get

$$\max_{y \in M} \{h_n(y, y)\} \leq D\epsilon + C_7 \left(\phi_G([n/(2B)]) + e^{-n/B} \right) .$$

Letting ϵ tend to zero yields the desired result, that is,

$$\Phi_M \preceq \phi_G .$$

6. Further remarks

It is natural to ask what role group invariance plays in Theorem 1.2. To be more precise, let $\mathcal{G}_i = (V_i, E_i)$ be two locally finite graphs with non-oriented edge sets. Let $N_i(x)$ be the number of neighbors of x in $\mathcal{G}_i$. Let $K_i(x, y) = 1/N_i(x)$ if $\{x, y\} \in E_i$ and $K_i(x, y) = 0$ otherwise. The Markov kernel K_i is reversible with respect to the measure N_i. In what follows we always assume that $\sup_{x \in V_i} N_i(x) < \infty$.

In this setting the question solved in Theorem 1.2 for Cayley graphs generalizes as follows. Assume that the graphs $\mathcal{G}_1$, $\mathcal{G}_2$ are quasi-isometric and let $\psi : V_1 \to V_2$ be a quasi-isometry.

1. Is it true that, as functions of n,

$$K_1^{2n}(x, x) \simeq K_2^{2n}(\psi(x), \psi(x)) \tag{6.1}$$

for any $x \in V_1$ (uniformly in x)?

2. Is it true that

$$\sup_{x \in V_1} K_1^{2n}(x, x) \simeq \sup_{y \in V_2} K_2^{2n}(y, y) \;? \tag{6.2}$$

In Cayley graphs, (6.1) and (6.2) are the same because, by invariance, $K_i^n(x, x) = K_i^n(\mathrm{id}, \mathrm{id})$. For general graphs, the argument of the present paper can be used to obtain a result weaker than (6.2), namely

$$\inf_{x \in V_1} K_1^{2n}(x, x) \leq C \sup_{y \in V_2} K_2^{2[n/C]}(y, y) \tag{6.3}$$

for some constant C. Of course, this proves (6.2) if

$$\inf_{x \in V_1} K_1^{2n}(x, x) \simeq \sup_{x \in V_1} K_1^{2n}(x, x) \tag{6.4}$$

and similarly for K_2. For instance, this extends the results of this paper to random walks that are quasi-transitive with respect to some group action.

Unfortunately, we do not know any satisfactory method to prove (6.4). To see how poorly this question is understood, consider the case where $\mathcal{G}_1$ is a Cayley graph of a group G with symmetric finite generating set S and $\mathcal{G}_2 = (G, E_2)$ where E_2 is obtained from E_1 by adding some edges between points at distance at most 10 of each other in $\mathcal{G}_1$. Then it is clear that $\mathcal{G}_1, \mathcal{G}_2$ are quasi-isometric. In fact, the identity map $G \to G$ is bi-Lipschitz in this case. Clearly, $\mathcal{G}_2$ is not necessarily a Cayley graph since we did not require invariance under the action of the group when adding edges. Even in this simple case we do not know how to prove that

$$\phi_S(n) \simeq \sup_{x \in G} K_2^{2n}(x, x) \tag{6.5}$$

or that

$$\inf_{x \in G} K_2^{2n}(x, x) \simeq \sup_{x \in G} K_2^{2n}(x, x) . \tag{6.6}$$

If G contains a polycyclic subgroup of finite index, then one can use Nash type inequalities and (6.3) to prove (6.5). See, e.g., [11]. Still it seems very reasonable to conjecture that (6.5) and (6.6) hold true on any group G with $\mathcal{G}_1$ and $\mathcal{G}_2$ as above.

These remarks and questions can of course be translated in the context of heat kernels on manifolds.

References

[1] Alexopoulos, G. A lower estimate for central probabilities on polycyclic groups, *Canadian Math. J.*, **44**, 897–910, (1992).

[2] Baldi, P., Lohoué, N., and Peyrière, J. Sur la classification des groupes récurrents, *CRAS*, Série I, **285**, 1103–1104, (1977).

[3] Bass, H. The degree of polynomial growth of finitely generated nilpotent groups, *Proc. London Math. Soc.*, **25**, 603–614, (1972).

[4] Brooks, R. Amenability and the spectrum of the Laplacian, *Bull. Am. Math. Soc.*, **6**, 87–89, (1982).

[5] Brown, L. and Kosaki, H. Jensen's inequality in semi-finite von Neumann algebras, *J. Operator Theory*, **23**, 3–19, (1990).

[6] Buser, P. A note on the isoperimetric constant, *Ann. Sci. École Norm. Sup.*, **15**, 213–230, (1982).

[7] Carlen, E., Kusuoka, S., and Stroock, D. Upper bounds for symmetric transition functions, *Ann. Inst. H. Poincaré, Prob. Stat.*, **23**, 245–287, (1987).

[8] Chavel, I. *Eigenvalues in Riemannian Geometry*, Academic Press, New York, 1984.

[9] Chavel, I. *Riemannian Geometry: A Modern Introduction*, Cambridge University Press, Cambridge, 1993.

[10] Cheeger, J., Gromov, M., and Taylor, M. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, *J. Differential Geometry*, **17**, 15–53, (1982).

[11] Coulhon, T. Ultracontractivity and Nash type inequalities, *J. Funct. Anal.*, 510–539, (1995).

[12] Coulhon, T. and Grigor'yan, A. On-diagonal lower bounds for heat kernels and Markov chains, *Duke Math. J.*, **89**, 133-199, (1997).

[13] Coulhon, T. and Saloff-Coste, L. Puissances d'un opérateur régularisant, *Ann. Inst. H. Poincaré, Prob. Stat.*, **26**, 419–436, (1990).

[14] Coulhon, T. and Saloff-Coste, L. Variétés riemanniennes isométriques à l'infini, *Rev. Math. Iberoamericana*, **11**, 687–726, (1995).

[15] Davies, E.B. *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics 92, Cambrige University Press, Cambridge, 1989.

[16] Følner, E. On groups with full Banach mean value, *Math. Scand.*, **3**, 243–254, (1955).

[17] Grigorchuk, R. On growth in group theory, in *Proc. Intl. Congress of Math.*, Kyoto, 1990, (1991).

[18] Gromov, M. Groups of polynomial growth and expanding maps, *Publ. Math. IHES*, **53**, 53–73, (1981).

[19] Gromov, M. *Asymptotic Invariants of Infinite Groups*, Geometric Group Theory, Vol. II, Niblo, G.A. and Roller, M.A., Eds., London Mathematical Society Lecture Note Series 182, Cambridge University Press, Cambridge, 1993.

[20] Gilbarg, D. and Trudinger, N. *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin, 1983.

[21] Horn, R. and Johnson, C. *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[22] Hebisch, W. and Saloff-Coste, L. Gaussian estimates for Markov chains and random walks on groups, *Ann. Prob.*, **21**, 673–709, (1993).

[23] Hunt, G. Some theorems concerning Brownian motion, *Trans. Am. Math. Soc.*, **81**, 294–319, (1955).

[24] Kanai, M. Rough isometries, and combinatorial approximations of geometries of non-compact riemannian manifolds, *J. Math. Soc. Japan*, **37**, 391–413, (1985).

[25] Kesten, H., Symmetric random walks on groups, *Trans. Am. Math. Soc.*, **92**, 336–354, (1959).

[26] Pittet, Ch. and Saloff-Coste, L. A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples, preprint, 1997.

[27] Pittet, Ch. and Saloff-Coste, L. Amenable groups, isoperimetric profiles and random walks, in *Geometric Group Theory Down Under*, Proceedings of a Special Year in Geometric Group Theory, Canberra, Australia, 1996, J. Cossey et al., Eds., Walter de Gruyker, Berlin-New York, 293–316, (1999).

[28] Pittet, Ch. and Saloff-Coste, L. Isoperimetry and random walk on discrete subgroups of connected Lie groups, in *Random Walk and Discrete Potential Theory*, Cortona, Cambridge University Press, Cambridge, 306–319, (1999).

[29] Saloff-Coste, L. Uniformly elliptic operators on Riemannian manifolds, *J. Diff. Geom.*, **36**, 417–450, (1992).

[30] Varopoulos, N. Brownian motion and transient groups, *Ann. Inst. Fourier*, **33**, 241–261, (1983).

[31] Varopoulos, N. Chaînes de Markov et inégalités isopérimétriques, *CRAS*, 298, série I, 233–235 & 465–468, (1984).

[32] Varopoulos, N. Théorie du potentiel sur les groupes nilpotents, *CRAS*, 301, série I, 143–144, (1985).

[33] Varopoulos, N. Théorie du potentiel sur des groupes et des variétés, *CRAS*, 302, série I, 203–205, (1986).

[34] Varopoulos, N. Analysis and geometry on groups, in *Proc. Intl. Congress of Math.*, Kyoto, 1990, (1991).

[35] Varopoulos, N., Saloff-Coste, L., and Coulhon, T. *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, (1992).

[36] Woess, W. Random walks on infinite graphs and groups—A survey on selected topics, *Bull. London Math. Soc.*, **26**, 1–60, (1994).

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