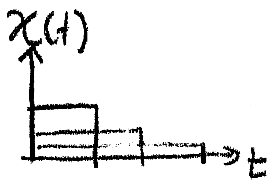
Math 424 - Assignment 1 - Solutions

Chapter 0, problem 8:

Let $x_n(t) = \begin{cases} 1/n & \text{on } [0,n] \\ 0 & \text{on } (n,\infty) \end{cases}$



then $x_n(t)$ converges uniformely to 0 as given any $\varepsilon > 0$, then for $1/n < \varepsilon$ i.e. $1/\varepsilon < n$

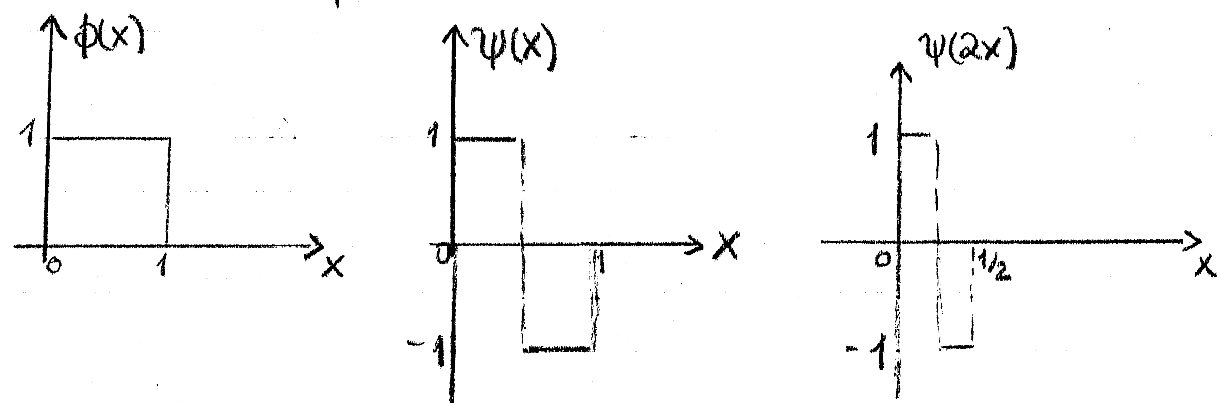
then $|x_n(t) - 0| = |1/n| < \varepsilon$ so that we have uniform convergence.

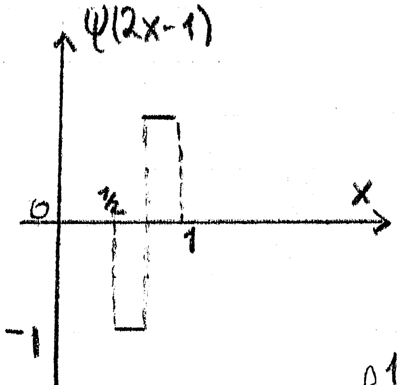
But $\int_0^\infty |x_n(t)|^2 dt = \int_0^n x_n(t)^2 dt = \int_0^n 1/n \, dt = 1 \quad \forall n$

$\Rightarrow x_n \not\to 0$ in $L^2[0,\infty)$

so the result 0.10 does not hold on $[0,\infty)$. $\square$

Chapter 0, problem 15:





You can actually immediately see from the graphs that the functions are mutually orthogonal.

E.g. let us compute more explicitly

$$\int_0^1 \psi(x)\cdot\psi(2x)\,dx = \int_0^{1/4} dx + \int_{1/4}^{1/2} -1\,dx = 0$$

or also immediately $\int_0^1 \psi(2x)\,\psi(2x-1)\,dx = 0$ as supports are disjoint.

Formally you should verify all orthogonality relations

Scaling $\psi(2x)$ by $\sqrt{2}$ & $\psi(2x-1)$ by $\sqrt{2}$ to get

$\psi_1(x) = \sqrt{2}\,\psi(2x)$ $\quad$ $\psi_2(x) = \sqrt{2}\cdot\psi(2x-1)$ we get an

orthonormal basis e.g. to check $\int_0^1 2\,\psi^2(2x)\,dx = 2\cdot\frac{1}{2} = 1$

Now simply apply Theorem 0.21 i.e. compute

$$\alpha_1 := \langle f, \phi\rangle = \int_0^1 x\,dx = \frac{1}{2} \qquad \alpha_2 := \langle f, \psi\rangle = \int_0^{1/2} x\,dx - \int_{1/2}^1 x\,dx$$

$$= \frac{1}{2}x^2\Big|_0^{1/2} - \frac{1}{2}x^2\Big|_{1/2}^1$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{4} - \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{4} = -\tfrac{1}{4}$$

$$\alpha_3 := \langle f, \psi_1 \rangle = \sqrt{2}\int_0^{1/4} x\,dx - \sqrt{2}\int_{1/4}^{1/2} x\,dx = \sqrt{2}\left(\tfrac{1}{2}\cdot\tfrac{1}{16} - \tfrac{1}{2}\cdot\tfrac{1}{4} + \tfrac{1}{2}\cdot\tfrac{1}{16}\right)$$

$$= \sqrt{2}/2\,(-1/8) = -\sqrt{2}/16$$

$$\alpha_4 := \langle f, \psi_2 \rangle = \sqrt{2}\cdot\int_{1/2}^{3/4} x\,dx - \sqrt{2}\cdot\int_{3/4}^{1} x\,dx$$

$$\frac{\sqrt{2}}{2}\left(x^2\Big|_{1/2}^{3/4} - x^2\Big|_{3/4}^{1}\right) = \sqrt{2}/2\,(9/16 - 1/4 - 1 + 9/16)$$

$$= -\frac{1}{8}\cdot(\sqrt{2}/2) = -\sqrt{2}/16$$

So the orthogonal projection $p(f)$ onto the space is

$$p(f) = \tfrac{1}{2}\phi(x) - \tfrac{1}{4}\psi(x) - \tfrac{1}{8}\psi(2x) - \tfrac{1}{8}\psi(2x-1)$$

$\square$

Chapter 0, problem 19 :

The defining equation for the adjoint is

$$\langle T(f), g\rangle = \langle f, T^*(g)\rangle \qquad f, g \in L^2(\mathbb{R})$$

Let us start with the LHS above

$$\langle T(f), g\rangle = \int_{\mathbb{R}} T(f)(x)\, \overline{g(x)}\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) K(x,y)\, dy\, \overline{g}(x)\, dx$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} f(y)\, \overline{g}(x)\, K(x,y)\, dy\, dx$$

$$= \iint_{B \subset \mathbb{R}^2} f(y)\, \overline{g}(x)\, K(x,y)\, dy\, dx$$

where B is the bounded set where K is supported i.e. $K(x,y)=0$ on B^c

conjugate symm.

$$= \iint_B \overline{g(x)\, \overline{K(x,y)}\, dx\, \overline{f(y)}\, dy}$$

- on bdd. sets we can interchange
- all fcts are integrable on B

e.g. $\int_a^b |g(x)|dx \leq |b-a|^{1/2} \int_a^b |g(x)|^2 dx < \infty$

Cauchy Schwarz

$$= \overline{\iint_{\mathbb{R}^2} g(x)\, \overline{K(x,y)}\, dx\, \overline{f(y)}\, dy}$$

$$= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} \overline{g(x)}\, K(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} \overline{\overline{K(x,y)}\, g(x)\, dx}\, dy = \langle f, T^*(g)\rangle$$

as defined in exercise

So since T^* as defined satisfies the adjoint equation, we are done. $\square$

Chapter 0, problem 21:

Suppose $\exists x \in V$ s.t. $Ax = b$. Assume for contradiction that there is also a vector $w \in W$ s.t. $A^*w = 0$ and $\langle b, w \rangle \neq 0$

i.e. $\langle Ax, w \rangle \neq 0$. Using the definiton of an adjoint we get

$$\langle x, A^*w \rangle \neq 0 \quad \text{but} \quad A^*w = 0 \Rightarrow \text{contradiction.}$$

Converseley, we need to show that if $Ax \neq b$ for any $x \in V$ then we can find $w \in W$ s.t. $A^*w = 0$ and $\langle b, w \rangle \neq 0$

Define $w = b - \underbrace{\text{projection}_{\text{range}A}(b)}_{:= p(b)} \neq 0$ as $b \neq Ax$ for any x

Also $\langle w, z \rangle = 0 \;\; \forall z \in \text{range}A$ by construction of the projection

But $(\text{range}A)^\perp = \ker(A^*)$ by problem 20 so $w \in \ker A^*$ i.e.

$$A^*w = 0$$

It remains to show $\langle b, w \rangle \neq 0$

If $\langle b, w \rangle = \langle b, b - p(b) \rangle = 0 \iff \langle b, b \rangle = \langle b, p(b) \rangle$

$$\|b\|^2 = \langle b, p(b) \rangle$$

But $\langle b, p(b) \rangle \overset{\text{C.S.}}{\leq} \|b\| \, \|p(b)\|$ so $\|p(b)\| = \|b\|$

But $b = b - p(b) + p(b)$ and $b - p(b) \in \text{range}(A)^\perp$, $p(b) \in \text{range}(A)$

so $\|b\|^2 = \|b - p(b)\|^2 + \|p(b)\|^2$

$> \|p(b)\|^2$ if the projection is non trivial $\Rightarrow$ ↯

Notice that so far we have implicitly assumed $b \neq 0$; but we need to look at this special case:

$b = 0 \Rightarrow \langle w, b \rangle = 0 \ \forall w$ i.e. we need to show that $\exists x$ s.t. $Ax = 0$ but $x = 0$ does the job. $\square$

Chapter 0, problem 23:

Suppose $\{\phi_i\}_{i\in I}$, I some index set are orthonormal in a vector space V and also assume (for contradiction) that

$$\sum_{k=1}^{n} c_{i_k}\phi_{i_k} = 0 \quad \text{for some } c_{i_k} \neq 0;$$

i.e that there exist a linear dependence relation.

Taking inner products on both sides and using bilinearity

$$\underset{\displaystyle 0}{\overset{}{\langle 0, \phi_{i_1}\rangle}} = \langle \sum_{k=1}^{n} c_{i_k}\phi_{i_k}, \phi_{i_1}\rangle = \sum_{k=1}^{n} c_{i_k}\langle \phi_{i_k}, \phi_{i_1}\rangle =$$

$$= c_{i_1}\langle \phi_{i_1}, \phi_{i_1}\rangle = c_{i_1} \neq 0 \Rightarrow \lightning$$

orthonormality

$\square$

Chapter 0, problem 26 :

(a) Unfortunately, this question is not as precise as it should. It is not defined what is meant by least squares problem.

Essentially the discussion of pages 23-25 gives that we search for a minimum of

$$\tilde{E} = \sum_{i=1}^{N} | y_i - (mx_i + b)|^2$$

Now this is equivalent to minimizing

$$E(m,b) = \sum_{i=1}^{N} | mx_i + b - y_i |^2 \quad \text{since } |a| = |-a| = |-1|\,|a|$$

(notice that we can assume $x_i, y_i \in \mathbb{R}$)

(b) Suppose $\frac{\partial E}{\partial b} = \frac{\partial E}{\partial m} = 0$ then E has a critical point

Since we have to compute the derivatives anyway we can try to use the calculus criterion for minima of multivariate functions, i.e.

$$E_{bb} E_{mm} - E_{mb}^2 > 0 \quad \text{and} \quad E_{mm} > 0$$

$\Rightarrow$ local minimum

$$E_b = \sum 2(mx_i + b - y_i) \qquad E_m = \sum 2(mx_i + b - y_i)\, x_i$$

$$E_{bb} = 2\sum_{i=1}^{N} 1 = 2N \qquad E_{mm} = 2\sum_{i=1}^{N} x_i^2 \Rightarrow E_{mm} > 0 \text{ if } x_i \neq x_j \text{ and } N \geq 2$$

$$E_{mb} = 2\sum x_i$$

$$E_{bb}E_{mm} - E_{mb}^2 = 4N \cdot \sum_{i=1}^{N} x_i^2 - \left(2\sum_{i=1}^{N} x_i\right)^2$$

$$= 4N \sum_{i=1}^{N} x_i^2 - 4\left(\sum_{i=1}^{N} x_i\right)^2$$

$$= 4\left(N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2\right)$$

$$= 4\left((N-1)\sum_{i=1}^{N} x_1^2 - \sum_{\substack{i,j\\ i\neq j}} x_i x_j\right) > 0$$

complete the square for $N \geq 2$, $x_i \neq x_j$ for some i,j

So we have a local minimum if we have a critical point $\Rightarrow$ global minimum.

[Alternatively, you can also use the convexity of $E(m,b)$ to get to the result...]

(c) $E_b = 0 \Leftrightarrow \sum 2(mx_i + b - y_i) = 0$

$E_m = 0 \Leftrightarrow \sum 2x_i(mx_i + b - y_i) = 0$

so that $\sum 2(mx_i - y_i) + Nb = 0$ i.e. $b = -\frac{\sum 2(mx_i - y_i)}{N}$

$$\Rightarrow 0 = \sum 2x_i\left(mx_i - \frac{\sum 2(mx_k - y_k)}{N} - y_i\right)$$

$$0 = \frac{1}{N}\sum 2x_i\left(Nmx_i - \sum 2(mx_k - y_k) - Ny_i\right)$$

$$0 = \sum_i 2Nmx_i^2 - \sum_i 2x_i \sum_k 2mx_k + \underbrace{\sum_i 2x_i \sum_k y_k - \sum_i Ny_i}_{=A}$$

$$0 = m\underbrace{\left(\sum_i 2Nx_i^2 - \sum_{i,k} 4x_i x_k\right)}_{=B}$$

$$\Rightarrow m = -A/B. \quad \square$$