Math 424 - Assignment 2 - Solutions

Chapter 1, problem 1:

$f(x)=x^2$, $f(-x)=(-x)^2=x^2$ so f is even and we know from Thm. 1.8. that $b_k=0\ \forall k$

$$a_0=\frac{1}{\pi}\int_0^\pi x^2\,dx = \tfrac{1}{3}\pi^2$$

integration by parts

$$a_k = \frac{2}{\pi}\int_0^\pi x^2\cos(kx)\,dx = \frac{2}{\pi}\left(\frac{x^2}{k}\sin(kx)\Big|_0^\pi - \int_0^\pi \frac{2x}{k}\cdot\sin(kx)\,dx\right)$$

$$= -\frac{4}{\pi k}\int_0^\pi x(\sin kx)\,dx = \frac{4}{\pi k}\left[\frac{x}{k}\cos kx\Big|_0^\pi + \int_0^\pi -\frac{1}{k}\cos(kx)\,dx\right]$$

$$= \frac{4\pi}{\pi k^2}\cos(k\pi) = \begin{cases} 4/k^2 & \text{if } k \text{ is even} \\ -4/k^2 & \text{if } k \text{ is odd}\end{cases}$$

so $$f(x) = \tfrac{1}{3}\pi^2 + \sum_{k=1}^{\infty}\frac{4}{k^2}(-1)^k\cdot\cos(kx)$$

This means we can immediately write the partial sums.

$$S_1 = \tfrac{1}{3}\pi^2 - 4\cos(x)$$

$$S_2 = \tfrac{1}{3}\pi^2 - 4\cos(x) + \cos(2x)$$

$$S_5 = \tfrac{1}{3}\pi^2 - 4\cos(x) + \cos(2x) - \tfrac{4}{9}\cos(3x) + \dots$$

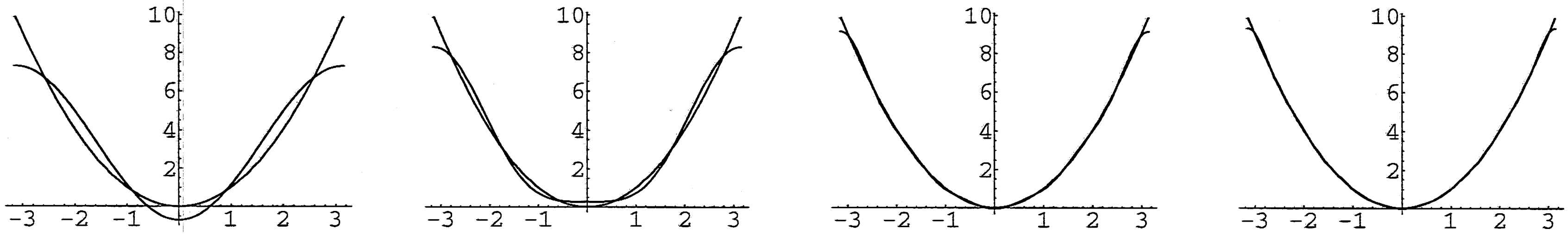
etc.

For the plots, see next page (it is OK if you use your favourite computer package to generate plots) $\longrightarrow$

We see that the partial sums work well on $[-\pi,\pi]$ and it is also not surprising from the theory that our coefficients from $[-\pi,\pi]$ ~~may~~ fail on $[-2\pi, 2\pi]$.

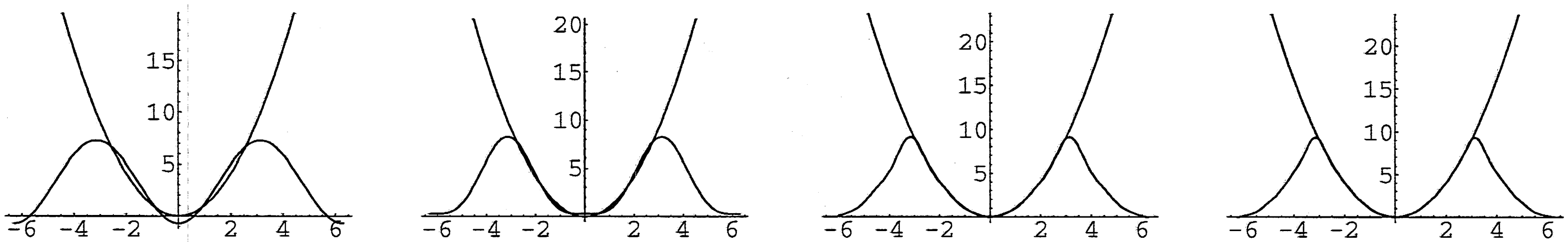
In[35]:= $f[x\_, N\_] := \dfrac{\pi \wedge 2}{3} + \sum_{k=1}^{N} \dfrac{4*(-1)\wedge k}{k \wedge 2} \mathrm{Cos}[k*x];$

```
p1 = Plot[{x^2, f[x, 1]}, {x, -π, π}, DisplayFunction → Identity];
p2 = Plot[{x^2, f[x, 2]}, {x, -π, π}, DisplayFunction → Identity];
p3 = Plot[{x^2, f[x, 5]}, {x, -π, π}, DisplayFunction → Identity];
p4 = Plot[{x^2, f[x, 7]}, {x, -π, π}, DisplayFunction → Identity];
Show[GraphicsArray[{p1, p2, p3, p4}]];
```



Now the same graphs for the interval $[-2\pi, 2\pi]$

In[46]:=
```
p5 = Plot[{x^2, f[x, 1]}, {x, -2*π, 2*π}, DisplayFunction → Identity];
p6 = Plot[{x^2, f[x, 2]}, {x, -2*π, 2*π}, DisplayFunction → Identity];
p7 = Plot[{x^2, f[x, 5]}, {x, -2*π, 2*π}, DisplayFunction → Identity];
p8 = Plot[{x^2, f[x, 7]}, {x, -2*π, 2*π}, DisplayFunction → Identity];
Show[GraphicsArray[{p5, p6, p7, p8}]];
```



2

Chapter 1, problem 3:

For a cosine series, we consider the even extension of $f(x)=x^2$, given by $f_e(x)=\begin{cases} f(x) & \text{if } x\in[0,\pi] \\ f(-x) & \text{if } x\in[-\pi,0) \end{cases}$

The discussion on p. 48 gives:

$$f(x)=a_0+\sum_{k=1}^{\infty} a_k\cos(kx) \qquad a_0=\frac{1}{\pi}\int_0^{\pi} x^2\,dx = \pi^2/3$$

$$a_k = 2/\pi\int_0^{\pi} x^2\cos(kx)\,dx = \begin{cases} 4/k^2 & \text{if } k \text{ is even} \\ -4/k^2 & \text{if } k \text{ is odd} \end{cases}$$

see problem 1

So $f(x)=\pi^2/3+\sum_{k=1}^{\infty} 4/k^2\cdot(-1)^k\cos(kx)$

Chapter 1, problem 4:

Again referring to page 48, we get using an odd extension of $f(x)=x^2$ that on $[0,1]$

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

where

Integration by parts

$$b_k = 2\int_0^1 x^2 \sin(k\pi x)\,dx = 2\left[-\frac{x^2}{k\pi}\cos(k\pi x)\Big|_0^1 + \int_0^1 \frac{2x}{k\pi}\cos(k\pi x)\,d\right.$$

$$= -2/_{k\pi}\cos(k\pi) + \frac{4}{k\pi}\cdot\int_0^1 x\cos(k\pi x)\,dx$$

i.b.p.

$$= -2/_{k\pi}\cos(k\pi) + 4/_{k^2\pi^2}(-1)\int_0^1 \sin(k\pi x)\,dx$$

$$= -2/_{k\pi}\cos(k\pi) + 4/_{k^3\pi^3}\cos(k\pi x)\Big|_0^1 = \left[-2/_{k\pi} + 4/_{k^3\pi^3}\right]\cos(k\pi) - 4/_{k^3\pi^3}$$

$$= \begin{cases} -2/_{k\pi} & \text{if } k \text{ is even} \\ -2/_{k\pi} - 8/_{k^3\pi^3} & \text{if } k \text{ is odd} \end{cases}$$

so $$f(x) = \sum_{k=1}^{\infty}\left(\left[-2/_{k\pi} + 4/_{k^3\pi^2}\right]\cos(k\pi) - 4/_{k^3\pi^3}\right)$$

Chapter 1, problem 7:

$f(x) = |\sin x| = |-\sin(x)| = |\sin(-x)|$ so f is even

Thm 1.8 gives that we just have to look at the cosine terms.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = -\tfrac{1}{\pi}\cos x \Big|_0^{\pi} = \tfrac{2}{\pi}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx \, dx \underset{(*)}{=} \dots = \frac{2(1+\cos k\pi)}{\pi - k^2\pi}$$

$$= \begin{cases} 0 & \text{if } k \text{ is odd } (k \neq 1) \\ 4/(\pi - k^2\pi) & \text{if } k \text{ is even} \end{cases}$$

so $$f(x) = \tfrac{2}{\pi} + \sum_{k=2}^{\infty} \left(\frac{2(1+\cos(k\pi))}{\pi - k^2\pi} \right) \cos(kx)$$

Notice if $k = 1 \Rightarrow a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx \underset{(*)}{=} \dots = 0$

(*) note that
$$\sin(mx+nx) = \sin(mx)\cos(nx) + \cos(mx)\sin(nx)$$
$$\sin(mx-nx) = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$$
so that
$$\sin(mx)\cos(nx) = \left(\sin((m+n)x) + \sin((m-n)x)\right)/2$$

which is the only non-trivial step in the integral.

Chapter 1 - problem 11:

$f(x) = \cos x$, we need an odd extension from $[0,\pi]$
so using the discussion on p. 48

$$f(x) = \sum_{k=2}^{\infty} b_k \sin(kx)$$

(*) previous problem, same trick

$$b_k = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(kx)\,dx \overset{(*)}{=} \dots = \frac{2k(1+\cos(k\pi))}{(k^2-1)\pi} \quad k \neq 1$$

again $b_1 = 0$

Chapter 1 - problem 16:

$f(x)$ cts $\quad f_e(x) = \begin{cases} f(x) & \text{on } [0,a] \\ f(-x) & \text{on } [-a,0) \end{cases} \rightarrow$ extend periodically to $\mathbb{R}$

at $x_0 = 0$; pick any sequence $x_k \to \bar{x}_0$ as $k \to \infty$ then

$$\lim_{k\to\infty} f_e(x_k) = \lim_{k\to\infty} f(-x_k) = \lim_{k\to\infty} f(\tilde{x}_k) = f(0)$$

$\tilde{x}_k = -x_k$, since $x_k \leq 0$

so f_e is cts at 0. Since $f_e(a) = f(-a)$ the periodic extension to $\mathbb{R}$ will also be continuous. $\square$

For odd periodic extensions, we can easily find a continuous f s.t. f cts on $[0,1]$ but $f_o = \begin{cases} f(x) & \text{on } [0,1] \\ -f(-x) & \text{on } [-1,0) \end{cases}$

is not. Simply take $f(x) = 1$ on $[0,1]$

$\Rightarrow f_o(x) = -1$ on $[-1,0)$
$\Rightarrow$ discts. at 0.

7

From the fact that for continuity we need

(1) $\quad -f(0) = f(0)$ we conclude $f(0) = 0$

also the same argument holds at a for $f(x)$ on $[0, a]$
so

(2) $\quad f(a) = 0$

$\Rightarrow$ then the periodic extension will be cts if f is.

$\square$