Math 424 - Assignment 3 - Solutions

Chapter 1, problem 15:

If we consider the periodic extension of f, say $\tilde{f}$, then

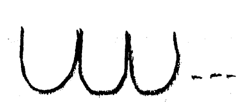
$$\begin{cases} \tilde{f}(k) = 0 \text{ if } k \in \mathbb{Z} \text{ \& } k \text{ is odd and } \tilde{f}(x) = 0 \text{ in some nbhd of } k \text{ identically.} \\ \tilde{f}(k) = 1 \text{ if } k \in \mathbb{Z} \text{ \& } k \text{ is even and } \tilde{f}(x) = 0 \text{ is some nbhd of } k \text{ identically} \end{cases}$$

So $F(x) = f(x)$ except at $x = \pm 1/2$ where $F(\pm 1/2) = 1/2$ using Theorem 1.28 (constants are obviously differentiable).

Chapter 1, problem 18:

To see that the previously calculated Fourier series converges uniformly

$$f(x) = 1/3\,\pi^2 + \sum_{k=1}^{\infty} 4/k^2 (-1)^k \cos(kx)$$

we could either observe that $f(x) = x^2$ on $[-\pi, \pi]$ has a piecewise smooth periodic extension and then use Theorem 1.30 (p. 72) 
or more explicitly observe

$$\sum_{k=1}^{\infty} |4/k^2 (-1)^k| = 4 \sum_{k=1}^{\infty} 1/k^2 < \infty$$

and then invoke Lemma 1.33 (p. 74)

Now since we have uniform convergence

$$\pi^2 = f(\pi) = \tfrac{1}{3}\pi^2 + \sum_{k=1}^{\infty} \tfrac{4}{k^2}(-1)^k \underbrace{\cos(k\pi)}_{=(-1)^k}$$

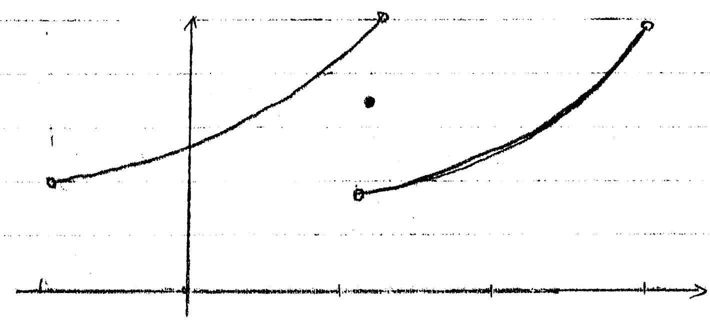
$$= \tfrac{1}{3}\pi^2 + \sum_{k=1}^{\infty} \tfrac{4}{k^2}$$

$$\Rightarrow \tfrac{2}{3}\pi^2 \cdot \tfrac{1}{4} = \pi^2/6 = \sum_{k=1}^{\infty} 1/k^2$$

$\square$

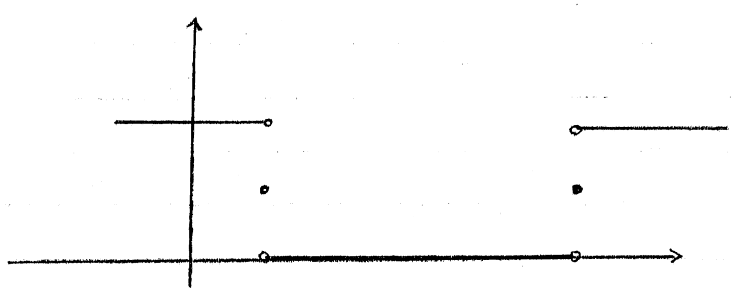
Chapter 1, problem 19: (Sketch of plots)

(a)



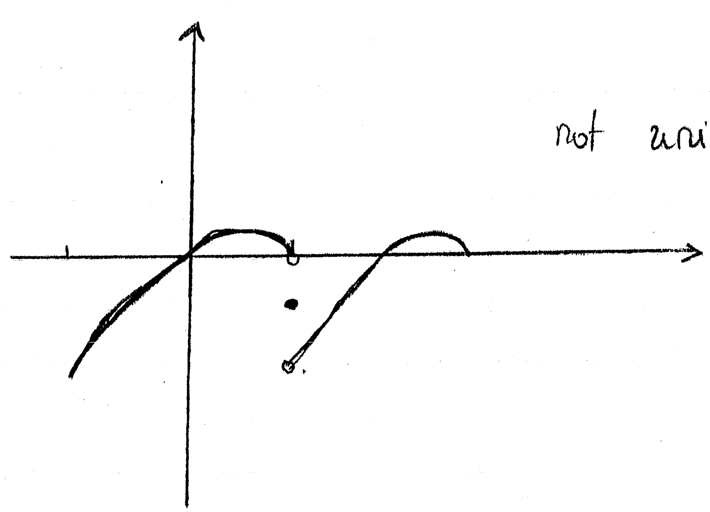
not uniform

(b)



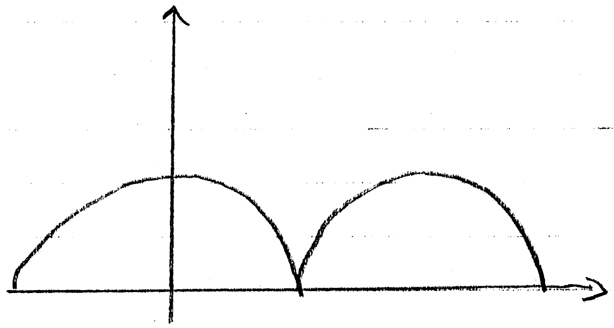
not uniform

(c)



not uniform

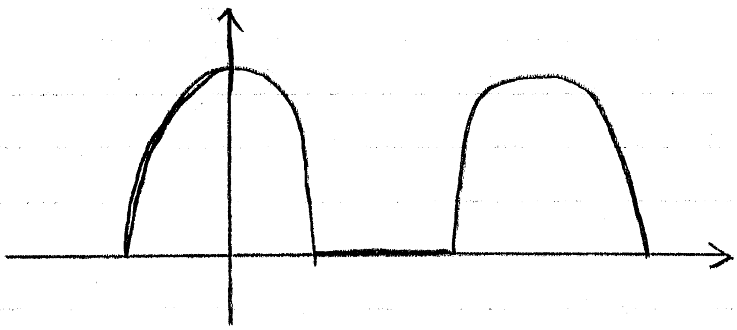
(d)



uniform (since $\tilde{f}$ is piecewise smooth)

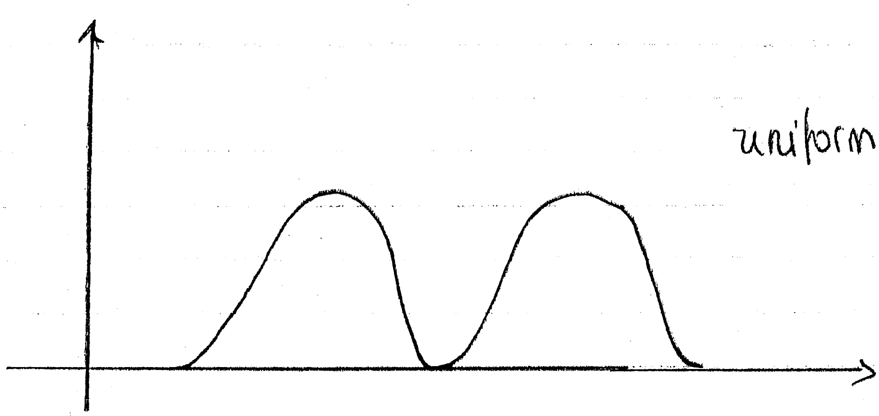
periodic extension of f

(e)

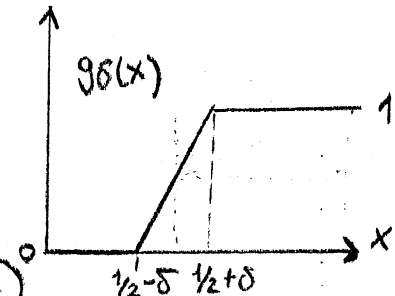


uniform ($\tilde{f}$ piecewise smooth)

(f)



uniform ($\tilde{f}$ piecewise smooth)



Chapter 1, problem 27:

$$\|f-g_\delta\|_{L^2}^2 = \int_0^1 |f-g_\delta|^2 dx = \int_{1/2-\delta}^{1/2+\delta} |f(x)-g_\delta(x)|^2 dx$$

$$= \int_{1/2-\delta}^{1/2} |0 - g_\delta(x)|^2 dx + \int_{1/2}^{1/2+\delta} |1-g_\delta(x)|^2 dx$$

$$= \underbrace{\int_{1/2-\delta}^{1/2} \left(\frac{x}{2\delta} - \frac{1}{4\delta} + \frac{1}{2}\right)^2 dx}_{\delta/12} + \underbrace{\int_{1/2}^{1/2+\delta} \left(-\frac{x}{2\delta} + \frac{1}{4\delta} + \frac{1}{2}\right)^2 dx}_{\delta/12}$$

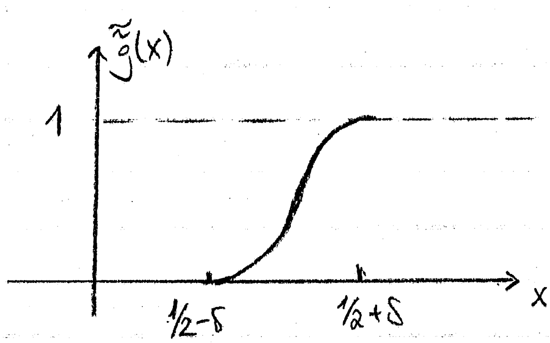
$$= \dots = \frac{\delta}{12} + \frac{\delta}{12} = \frac{\delta}{6} \longrightarrow 0 \text{ as } \delta \to 0$$

□

Yes we can modify g_δ to make it C^1 and still converge in L^2 to f; e.g. let

$$\tilde{g}_\delta(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2-\delta) \\ \frac{1}{2}\left[\sin\left(\frac{\pi(x-1/2)}{2\delta}\right)+1\right] & \text{if } x \in [1/2-\delta, 1/2+\delta) \\ 1 & \text{if } x \in [1/2+\delta, 1] \end{cases}$$

Notice that $\tilde{\tilde{g}}(x) := \frac{1}{2}\left[\sin\left(\frac{\pi(x-1/2)}{2\delta}\right)+1\right]$ looks like



i.e. is a smooth function "connecting" from 0 to 1.

We can e.g. directly verify:

$$\tilde{\tilde{g}}(1/2-\delta)=0 \quad \tilde{\tilde{g}}(1/2+\delta)=1$$

Also $\tilde{\tilde{g}}'(x) = \pi/4\delta \cdot \cos\left(\frac{\pi(x-1/2)}{2\delta}\right)$

so that $\tilde{\tilde{g}}'(1/2-\delta) = 0 = \tilde{\tilde{g}}'(1/2+\delta)$ i.e. we "join C^1"

For convergence, consider direct integration

$$\|f-g_\delta\|_2^2 = \int_{1/2-\delta}^{1/2} (\tilde{g}_\delta(x))^2\,dx + \int_{1/2}^{1/2+\delta} (1-\tilde{g}_\delta(x))^2\,dx = \ldots$$

$$= (3/8 - 1/\pi)\delta + (3/8 + 1/\pi)\delta \to 0 \text{ as } \delta \to 0$$

$\square$

Remark: I have omitted the details of 2 integrations above, both are easy and involve only a polynomial respectively a sine and I hope you can do these without problems.

If you can not please talk to me and I can help you on revision.

Chapter 1, problem 28:

(a) Considering a periodic extension, f is odd and so only sine terms appear in the Fourier series

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin(kx)\,dx$$

$$= 2/\pi\left[\int_0^{\pi}\pi\sin(kx)\,dx - \int_0^{\pi}x\sin(kx)\,dx\right]$$

$$\overset{\text{i.b.p.}}{=} -2\cdot\frac{1}{k}\cos(kx)\Big|_0^{\pi} - \frac{2}{\pi}\Big[-\frac{1}{k}\cos(kx)\,x\Big|_0^{\pi} - \underbrace{\int_0^{\pi}\frac{1}{k}\cos(kx)\,dx}_{=0}\Big]$$

$$= -2/k\,[\cos k\pi - 1] + \frac{2}{k}\cos(k\pi) = 2/k \qquad \square$$

(b) Define

$$g_N(x) = 2\cdot\sum_{n=1}^{N}\frac{\sin nx}{n} - (\pi - x)$$

(c)

$$g_N'(x) = 2\Big(\sum_{n=1}^{N}\cos nx\Big) + 1 \qquad \frac{d}{dx}\Big(\frac{\sin nx}{n}\Big) = \cos nx$$

$$= 2\pi\cdot\Big(\frac{1}{\pi}\Big(\frac{1}{2} + \cos x + \cos 2x + \dots + \cos Nx\Big)\Big)$$

$$= 2\pi\, P_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)} \quad \text{by Lemma 1.23.}$$

$\square$

(d)

$$g_N'(x) \overset{!}{=} 0 \quad \text{i.e.} \quad \frac{\sin((N+1/2)x)}{\sin(x/2)} = 0$$

$$\text{iff} \quad \sin((N+1/2)x) = 0 \quad \text{if } x \neq 0 \text{ and } x > 0$$

the first zero of sine occurs at π to the right of zero so

$$(N+1/2)x = \pi \implies x = \pi/(N+1/2) \qquad \square$$

(e) $$g_N(\theta_N) = \int_0^{\theta_N} g_N'(x)\,dx + g_N(0) \quad \text{by FTC}$$

so
$$g_N(\theta_N) \overset{(c)}{=} \int_0^{\theta_N} \frac{\sin((N+1/2)x)}{\sin(x/2)}\,dx - \pi$$

since $g_N(0) = -\pi$. $\square$

(f) $$\int_0^{\theta_N} \frac{\sin(N+1/2)x}{\sin x/2}\,dx \underset{\phi=(N+1/2)x}{=} \int_0^{\pi} \frac{\sin\phi}{\sin(\phi/(2N+1))}(N+1/2)^{-1}\,d\phi$$

Now $$\lim_{N\to\infty} g_N(\theta_N) = \lim_{N\to\infty} \int_0^{\pi} \frac{1/(N+1/2)}{\sin(\phi/(2N+1))} \sin\phi\,d\phi - \pi$$

$$= \lim_{N\to\infty} \int_0^{\pi} \frac{2}{\phi}\sin\phi \cdot \left(\frac{\phi/(2N+1)}{\sin \phi/(2N+1)}\right) d\phi - \pi$$

$$= 2\int_0^{\pi} \frac{\sin\phi}{\phi} \underbrace{\lim_{N\to\infty}\left(\frac{\phi/(2N+1)}{\sin(\phi/(2N+1))}\right)}_{=1 \text{ since } \sin t/t \to 1 \text{ as } t\to 0} d\phi - \pi = 2\cdot\int_0^{\pi} \frac{\sin x}{x}\,dx - \pi \quad \square$$

Notice: We have interchanged limits here; this is a nontrivial operation, but in this case justified. A course in integration theory is going to tell you why.

(g) see attached printout.

\9

Here is the numerical integration for the function in problem 28 (g)

```
In[61]:= NIntegrate[2 * Sin[x] / x, {x, 0, π}] - π

Out[61]= 0.562281
```

Although it is only one line in this case and you might be justified in not providing such a printout please notice that it is in general good practice to attach a printout to the homework of the code you used in a software like Mathematica, Maple, Matlab et. al.