Math 424 - Assignment 6 - Solutions

Problem 2, Chap. 3:

(1)
$$\mathcal{F}[z]_j = \sum_{m=0}^{n-1} z_m e^{-2\pi i j m/n} = \sum_{m=0}^{n-1} y_{m+1} e^{-2\pi i j m/n}$$
$$= \left(\sum_{m=0}^{n-1} y_{m+1} e^{-2\pi i j (m+1)/n}\right) e^{2\pi i/n}$$
$$= \left(\sum_{m=1}^{n} y_m e^{-2\pi i j m/n}\right) e^{2\pi i/n}$$
$$= \left(\sum_{m=0}^{n-1} y_m e^{-2\pi i j m/n}\right) e^{2\pi i/n} \quad \text{since } y_0 = y_n \text{ \& } e^{2\pi i 0/n} = e^{2\pi i n/n}$$

$\square$ (shifts and translations)

(3)
$$\mathcal{F}[y * z]_k = \sum_{m=0}^{n-1} [y*z]_m e^{-2\pi i k m/n}$$
$$= \sum_{m=0}^{n-1} \sum_{j=0}^{n-1} y_j z_{m-j} e^{-2\pi i k m/n} = \sum_{m=0}^{n-1} \sum_{j=0}^{n-1} y_j e^{-2\pi i k j/n} \underbrace{z_{m-j}\,\cdot}_{\exp(-2\pi i k \frac{m}{n} + 2\pi i k j/n)}$$
$$= \underbrace{\sum_{j=0}^{n-1} y_j e^{-2\pi i k j/n}}_{\mathcal{F}[y]_k} \underbrace{\sum_{m=0}^{n-1} z_{m-j} e^{-2\pi i k(m-j)}}_{= \mathcal{F}[z]_k \text{ by periodicity of } z}$$

$\square$ (convolution)

(4)
$$\mathcal{F}[y]_{n-k} = \sum_{j=0}^{n-1} y_j \overline{\omega}^{j(n-k)} = \sum_{j=0}^{n-1} y_j \overline{\omega}^{-jk} = \overline{\mathcal{F}[y]_k}$$

since $\overline{y_j} = y_j$

$\square$

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(2) Notice we should also show that

$$[y * z]_k \in S_n \quad \text{i.e}$$

$$\begin{array}{ccc} [y * z]_{k+n} & = & [y * z]_k \\ \| & & \| \\ \sum_{j=0}^{n-1} y_j z_{k+n-j} & = & \sum_{j=0}^{n-1} y_j z_{k-j} \end{array} \qquad \text{as } z \in S_n$$

under exp here means

$$e^{2\pi i \frac{PM}{N}\lambda e} = e^{2\pi i \lambda e} = e^{s \cdot 2\pi i} = 1 \quad s \in \mathbb{Z}$$

so that we get

$$\frac{1}{N}\sum_{n=0}^{N-1} \exp(-2\pi i(t_{q/T}(\lambda M+\mu)))\exp(-2\pi i(n(\mu+\lambda M)/N))$$

as required. $\square$

(c) Essentially we us induction here on m

For $m=2$ we have $N = P_1 P_2$

Now we solve $P_1 + P_2$ smaller interpolation problems and need

$$\underbrace{P_1 P_2^2 + P_2 P_1^2}_{\shortparallel} \text{ operations}$$
$$N(P_1 + P_2)$$

Now we simply consider $N = P_1 \cdots P_{m-1} P_m$

$$\underbrace{(P_1 P_2 \cdots P_{m-1})^2}_{} P_m + P_m^2 (P_1 \cdots P_{m-1}) \text{ ops}$$

by induction

$$= P_1 \cdots P_{m-1}(P_1 + P_2 + \ldots + P_{m-1}) + P_m^2 (P_1 \cdots P_{m-1}) \text{ ops}$$

$$= N(P_1 + \ldots + P_m) \text{ ops}$$

$\square$

Problem 6.24. from handout:

(a) Let's compute $y(t_m) =$

$$= \sum_{k=0}^{N-1} e^{-2\pi i k t_0/T} \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i k n/N} y_n e^{2\pi i k t_m/T}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} e^{-2\pi i k t_0/T} e^{-2\pi i k n/N} e^{2\pi i k t_m/T} y_n$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} e^{-2\pi i k(-t_0 - mT/N + t_0)/T} e^{-2\pi i k n/N} y_n$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi i k m/N - 2\pi i k n/N} y_n \Rightarrow$$

if $m = n \Rightarrow \frac{1}{N} \sum_{k=0}^{N-1} y_m = N/N \, y_m = y_m$

if $m \neq n \quad \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k/N (m-n)} y_n = \frac{1}{N} \frac{1 - e^{2\pi i N/N \overbrace{(m-n)}^{\neq 0}}}{1 - (\ldots)} = 0$

so we indeed get $y(t_m) = y_m$

(b) (i) We want $y_r(t_{r+\ell P}) = \sum_{\mu=0}^{M-1} c_\mu^{(r)} e^{2\pi i \mu (t_{r+\ell P})/T} \stackrel{!}{=} y_{r+\ell P}$

we have a uniformly space grid $t_{r+\ell P} = t_r + \ell P/N$

so that $c_\mu^{(r)} = e^{-2\pi i \mu t_r/T} \frac{1}{M} \sum_{\ell=0}^{M-1} e^{-2\pi i \mu \ell/M} y_{r+\ell P}$

Checking this is the same procedure as in (a) of course.

(b) (ii) Again we just need to replace suitable indices in our result for (a), namely $N \to P$ $T \to T/M$ $k \to \lambda$

$$C_{\mu\lambda} = e^{-2\pi i \lambda t_0 M/T} \frac{1}{P} \sum_{n=0}^{P-1} e^{-2\pi i \lambda n/P} C_\mu^{(n)}$$

(b) (iii) If we consider c_k from (a) and set $k = \mu + \lambda M$ we get

$$c_k = e^{-2\pi i(\mu+\lambda M)t_0/T} \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i(\mu+\lambda M)n/N} Y_n$$

We need to show that this equals

$$C_{\mu\lambda} \underset{\text{by (ii)}}{=} e^{-2\pi i \lambda t_0 M/T} \frac{1}{P} \sum_{r=0}^{P-1} e^{-2\pi i \lambda r/P} C_\mu^{(r)}$$

now using (ii) this equals

$$e^{-2\pi i \lambda t_0 M/T}\, 1/P \sum_{r=0}^{P-1} e^{-2\pi i \lambda r/P} e^{-2\pi i \mu t_r/T} \frac{1}{M} \sum_{\ell=0}^{M-1} e^{-2\pi i \mu \ell/M} Y_{r+\ell P}$$

$t_r = t_0 + rT/N$

$$= \frac{1}{\underset{N}{PM}} \sum_{r=0}^{P-1} \sum_{\ell=0}^{M-1} \exp\left(-2\pi i\left[\lambda t_0 M/T + \lambda r/P + \mu t_0/T + rT/N \cdot \mu/T + \mu\ell/M\right]\right) Y_{r+\ell P}$$

we already see that $r+\ell P$ varies between 0 and $(P-1)+(M-1)P = MP-1 = N-1$, so that the summation is correct except that we have to simplify the elements in the exponential, the "t_0"-terms give

$$t_0\left(\lambda M/T + \mu/T\right) = t_0/T(\lambda M + \mu) \quad \text{as required.}$$

The rest gives $\lambda r/P + r/N \mu + \mu\ell/M + Mr/P \overset{\text{under exp}}{=}$

$$\frac{\lambda PM\ell}{N} + \left(M\lambda r + \underbrace{r\mu + \mu\ell P}_{(r+\ell P)\mu}\right)/N = \frac{(r+\ell P)\cdot(\mu+\lambda M)}{N}$$