Math 424 – Assignment 8 – Solutions

Chapter 5, problem 10:

(a) $\sqrt{2\pi}\,\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt$

$$= \int_0^1 (1-t)\, e^{-it\xi}\, dt + \int_{-1}^0 (1+t)\, e^{-it\xi}\, dt$$

$$= \int_0^1 e^{-it\xi}\, dt - \int_0^1 t\, e^{-it\xi}\, dt + \int_{-1}^0 e^{-it\xi}\, dt + \int_{-1}^0 t\, e^{-it\xi}\, dt$$

$$= -\frac{1}{i\xi} e^{-it\xi}\Big|_{-1}^{1} - \left[-\frac{t}{i\xi} e^{-it\xi}\Big|_0^1 + \int_0^1 \frac{1}{i\xi} e^{-it\xi}\, d\xi \right] + \left[-\frac{t}{i\xi} e^{-it\xi}\Big|_{-1}^0 + \int_{-1}^0 \frac{1}{i\xi} e^{-it\xi}\, dt \right]$$

$$= \cancel{-\frac{1}{i\xi} e^{-i\xi}} + \cancel{\frac{1}{i\xi} e^{i\xi}} + \frac{1}{i\xi} e^{-i\xi} - \frac{1}{\xi^2} e^{-it\xi}\Big]_0^1 - \cancel{\frac{e^{i\xi}}{i\xi}} + \frac{1}{\xi^2} e^{-it\xi}\Big]_{-1}^0$$

$$= -\frac{1}{\xi^2} e^{-i\xi} + \frac{1}{\xi^2} + \frac{1}{\xi^2} - \frac{1}{\xi^2} e^{i\xi} = \frac{1}{\xi^2}\left[-e^{-i\xi} - e^{i\xi} \right] + \frac{2}{\xi^2}$$

$$= -\frac{1}{\xi^2}\left[\cos\xi - i\sin\xi + \cos\xi + i\sin\xi - 2 \right] = \frac{-2\cos\xi + 2}{\xi^2} = (*)$$

$\frac{1-\cos x}{2} = \sin^2(x/2)$ can be used to get

$$(*) = 2 \cdot 2 \cdot \sin^2(\xi/2) / \xi^2$$

$$\Rightarrow \hat{f}(\xi) = 2 \cdot \sqrt{\frac{2}{\pi}}\, \frac{\sin^2(\xi/2)}{\xi^2} \qquad \square$$

(b) (5.28.) reads $\csc^2(\xi/2) = \sum_{k\in\mathbb{Z}}' \frac{4}{(\xi+2\pi k)^2}$

Differentiating twice gives

$$\frac{d}{d\xi}\left(-\cot(\xi/2)\csc^2(\xi/2)\right) = \frac{d}{d\xi}\sum_{k\in\mathbb{Z}}' \frac{4\cdot(-2)}{(\xi+2\pi k)^3}$$

$$\Leftrightarrow \quad \cot^2(\xi/2)\csc^2(\xi/2) + \frac{1}{2}\csc^4(\xi/2) = \sum_{k\in\mathbb{Z}}' \frac{4\cdot 6}{(\xi+2\pi k)^4} := A$$

$$\Leftrightarrow \quad \csc^2(\xi/2)\left(\cot^2(\xi/2) + 1/2\csc^2(\xi/2)\right) = A$$

$$\Leftrightarrow \quad \frac{1}{\sin^2(\xi/2)}\left(\frac{\cos^2(\xi/2)}{\sin^2(\xi/2)} + \frac{1/2}{\sin^2(\xi/2)}\right) = A$$

$$\Leftrightarrow \quad \frac{1}{\sin^4(\xi/2)}\left(1 - \sin^2(\xi/2) + 1/2\right) = A$$

$$\Leftrightarrow \quad \frac{3 - 2\sin^2(\xi/2)}{48\sin^4(\xi/2)} = \sum_{k\in\mathbb{Z}} \frac{1}{(\xi+2\pi k)^4} \quad \square$$

(c) Thm 5.18. will give the result if we can show that

$$\sum_{k\in\mathbb{Z}}' |\hat{\phi}(\xi+2\pi k)|^2 = 1/2\pi$$

$$\|$$

$$\sum_{k\in\mathbb{Z}}' \left| 2\sqrt{\frac{2}{\pi}} \frac{\sin^2\left(\frac{\xi+2\pi k}{2}\right)}{(\xi+2\pi k)^2\sqrt{1-2/3\sin^2\left(\frac{\xi+2\pi k}{2}\right)}}\right|^2$$

$$= \frac{4\cdot 2}{\pi} \sum_{k\in\mathbb{Z}} \frac{\sin^4\left(\frac{\xi+2\pi k}{2}\right)}{(\xi+2\pi k)^4\left(1-\frac{2}{3}\sin^2\left(\frac{\xi+2\pi k}{2}\right)\right)}$$

$$= 3\cdot \frac{8}{\pi} \sum_{k\in\mathbb{Z}} \frac{\sin^4\left(\xi/2+\pi k\right)}{(\xi+2\pi k)^4\left(3-2\underbrace{\sin^2(\xi/2+\pi k)}_{=\sin^2(\xi/2)}\right)}$$

just changes sign by periodicity and $(\ )^2$ makes this sign 1

$$= \frac{24}{\pi}\left(\sum_{k\in\mathbb{Z}} \frac{1}{(\xi+2\pi k)^4}\right)\cdot \frac{\sin^4(\xi/2)}{3-2\sin^2(\xi/2)}$$

$$= \frac{24}{\pi}\cdot\frac{1}{48} = \frac{1}{2\pi}$$

using (b)

as required to apply 5.18. □

Chapter 5, problem 11:

We have to verify $P(1)=1$, $|P(z)|^2+|P(-z)|^2=1$ for $|z|=1$

$$|P(e^{it})|>0 \quad \text{for } |t|\leq \pi/2$$

$$P(1)=\tfrac{1}{2}\sum_{k=0}^{3} p_k = \tfrac{1}{2}\left[\frac{1+\sqrt{3}}{4}+\frac{3+\sqrt{3}}{4}+\frac{3-\sqrt{3}}{4}+\frac{1-\sqrt{3}}{4}\right]$$

$$=\frac{1}{8}\cdot[1+3+3+1]=1 \quad \square$$

For the others two parts we can use a CAS;

see relevant printout on next page (Mathematica)

```
In[6]:= p0 = (1 + Sqrt[3]) / 4;
        p1 = (3 + Sqrt[3]) / 4;
        p2 = (3 - Sqrt[3]) / 4;
        p3 = (1 - Sqrt[3]) / 4;
        P[z_] := 1 / 2 * (p0 + p1 * z + p2 * z ^ 2 + p3 * z ^ 3)
```

The next command loads a package to symbolically declare the type of variables (e.g. real or complex). We declare t to be real–valued and put in z=Exp[*i**t] to represent a complex number with unit modulus.

```
In[42]:= << Algebra`ReIm`

In[75]:= t /: Im[t] = 0;
         A = Re[P[Exp[i * t]]] ^ 2 + Im[P[Exp[i * t]]] ^ 2
         B = Re[P[-Exp[i * t]]] ^ 2 + Im[P[-Exp[i * t]]] ^ 2
```

Out[76]= $$\left(\frac{1}{8}+\frac{\sqrt{3}}{8}+\frac{3\,\mathrm{Cos}[t]}{8}+\frac{1}{8}\sqrt{3}\,\mathrm{Cos}[t]+\frac{3}{8}\mathrm{Cos}[2\,t]-\frac{1}{8}\sqrt{3}\,\mathrm{Cos}[2\,t]+\frac{1}{8}\mathrm{Cos}[3\,t]-\frac{1}{8}\sqrt{3}\,\mathrm{Cos}[3\,t]\right)^2+\left(\frac{3\,\mathrm{Sin}[t]}{8}+\frac{1}{8}\sqrt{3}\,\mathrm{Sin}[t]+\frac{3}{8}\mathrm{Sin}[2\,t]-\frac{1}{8}\sqrt{3}\,\mathrm{Sin}[2\,t]+\frac{1}{8}\mathrm{Sin}[3\,t]-\frac{1}{8}\sqrt{3}\,\mathrm{Sin}[3\,t]\right)^2$$

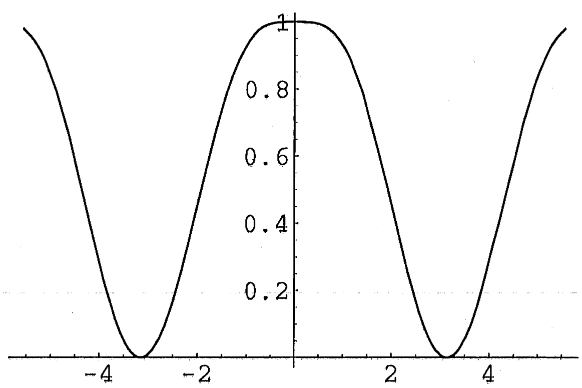
Out[77]= $$\left(\frac{1}{8}+\frac{\sqrt{3}}{8}-\frac{3\,\mathrm{Cos}[t]}{8}-\frac{1}{8}\sqrt{3}\,\mathrm{Cos}[t]+\frac{3}{8}\mathrm{Cos}[2\,t]-\frac{1}{8}\sqrt{3}\,\mathrm{Cos}[2\,t]-\frac{1}{8}\mathrm{Cos}[3\,t]+\frac{1}{8}\sqrt{3}\,\mathrm{Cos}[3\,t]\right)^2+\left(-\frac{3\,\mathrm{Sin}[t]}{8}-\frac{1}{8}\sqrt{3}\,\mathrm{Sin}[t]+\frac{3}{8}\mathrm{Sin}[2\,t]-\frac{1}{8}\sqrt{3}\,\mathrm{Sin}[2\,t]-\frac{1}{8}\mathrm{Sin}[3\,t]+\frac{1}{8}\sqrt{3}\,\mathrm{Sin}[3\,t]\right)^2$$

```
In[70]:= Simplify[A + B]

Out[70]= 1
```

And that is precisely what we had to prove. Please note that if you use a Computer Algebra System (CAS) you HAVE TO supply a printed version of the commands you used. Now for the plot of the second part we get:

```
In[74]:= Plot[Abs[P[Exp[i * x]]], {x, -π / 2 - 4, π / 2 + 4}]
```




```
Out[74]= - Graphics -
```

So we are clearly positive inside the interval [–2,2] and since $\pi/2<2$ we get the result.

Chapter 5, problem 13:

see p. 213, it should read

$$P(z) = 1/2 \sum_{k \in \mathbb{Z}} p_k z^k$$

Now define $Q(z) = -z\,\overline{P(-z)}$

$|z| = 1$ so set $z = e^{it}$ $t \in \mathbb{R}$ then we get $-z\,\overline{P(-z)} =$

$$-\frac{e^{it}}{2} \overline{\sum_{k \in \mathbb{Z}} p_k (-e^{it})^k} = e^{it}/2 \sum_{k \in \mathbb{Z}} \overline{p_k}\, e^{-itk} (-1)^{1-k}$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z}} e^{(1-k)it}\, \overline{p_k} \underset{m=1-k}{=} \frac{1}{2} \sum_{m \in \mathbb{Z}} e^{mit}\, \overline{p_{1-m}} (-1)^m$$

$\square$

Chapter 5, problem 14:

Thm: Suppose ϕ satisfies $\int \phi(x-k)\phi(x-\ell)\,dx = \delta_{k\ell}$ and $\phi(x) = \sum_k p_k \phi(2x-k)$; set $\psi(x) = \sum_k q_k \phi(2x-k)$ and let $Q(z) = \sum_k q_k z^k$ then TFAE

(1) $\int \psi(x-k)\phi(x-\ell)\,dx = 0 \quad \forall k,\ell \in \mathbb{Z}$

(2) $P(z)\overline{Q(z)} + P(-z)\overline{Q(-z)} = 0 \quad \forall z$ st. $|z|=1$

Proof: First we use Thm 5.18. which gives that (1) $\Leftrightarrow$ we have

$$\sum_{k\in\mathbb{Z}} \hat{\phi}(\xi+2\pi k)\overline{\hat{\psi}(\xi+2\pi k)} = 0 \quad (3)$$

since if $\psi(x)$ is orthogonal to $\phi(x-\ell)$ $\forall \ell \in \mathbb{Z}$ this means

$$\int \phi(x-\ell)\psi(x)\,dx = 0$$

$x = z-m$ gives $\int \phi(z-m-\ell)\psi(x-m)\,dx = 0$

so that indeed (1) iff (3).

So if we can construct a calculation which shows that (3) $\Leftrightarrow$ (1) we are done

$$\sum_{k\in\mathbb{Z}} \hat{\phi}(\xi+2\pi k)\overline{\hat{\psi}(\xi+2\pi k)} = 0$$

By theorem 5.19. we have

$$\begin{cases} \hat{\phi}(\tilde{\xi}) = \hat{\phi}(\tilde{\xi}/2)\, P(e^{-i\tilde{\xi}/2}) \\ \hat{\psi}(\tilde{\xi}) = \hat{\phi}(\tilde{\xi}/2)\, Q(e^{-i\tilde{\xi}/2}) \end{cases}$$

(same argument as in proof to 5.19)

So

$$\sum_{l\in\mathbb{Z}} \hat{\phi}(\xi + 2\pi(2l))\, \overline{\hat{\psi}(\xi + 2\pi(2l))} + \sum_{l\in\mathbb{Z}} \hat{\phi}(\xi + 2\pi(2l+1))\, \overline{\hat{\psi}(\xi + 2\pi(2l+1))} = 0$$

$$\Leftrightarrow \sum_{l\in\mathbb{Z}} \hat{\phi}(\xi/2 + 2\pi l)\, \overline{\hat{\phi}(\xi/2 + 2\pi l)} \cdot P(e^{-i\xi/2})\, \overline{Q(e^{-i\xi/2})}$$

$e^{2\pi i} = 1$

$e^{-i\pi} = -1$

$$+ \sum_{l\in\mathbb{Z}} \hat{\phi}(\xi/2 + \pi(2l+1))\, \overline{\hat{\phi}(\xi/2 + \pi(2l+1))}\, P(-e^{-i(\xi/2)})\, \overline{Q(-e^{-i\xi/2})} = 0$$

$$\Leftrightarrow \tfrac{1}{2\pi} P(e^{-i\xi/2})\, \overline{Q(e^{-i\xi/2})} + \tfrac{1}{2\pi} P(-e^{-i\xi/2})\, \overline{Q(-e^{-i\xi/2})} = 0$$

using Thm 5.18 with $\xi/2 + 2\pi l$ & $(\xi/2 + \pi) + 2\pi l$

Therefore the result follows if we multiply the last equation by 2π & realize that ξ was arbitrary so that $e^{i\xi/2}$ gives all elements in $\mathbb{C}$ s.t. $|\cdot| = 1$.

$\square$

Chapter 5, problem 17:

Let us explicitly calculate the support ...

Suppose $p_k = 0$ for $-N \leq k \leq M$ $N, M \in \mathbb{N}_0$

$$\phi_n(x) = \sum_{k \in \mathbb{Z}} p_k \phi_{n-1}(2x-k) \qquad \text{from p.217 (5.34)}$$

$$= \sum_{-N \leq k \leq M} p_k \phi_{n-1}(2x-k) = \sum_{-N \leq k \leq M} p_k \sum_{-N \leq m \leq M} p_m \phi_{n-2}(2(2x-k)-m)$$

$$= \sum_{\substack{-N \leq k \leq M \\ m}} p_k p_m \phi_{n-2}(4x-2k-m)$$

$$= \sum_{-N \leq k, m, l \leq M} p_k p_m p_l \phi_{n-3}(\underbrace{2(4x-2k-m)-l}_{=8x-4k-2m-l}) \qquad \text{etc.}$$

Using a more general indexing we can write this as

$$\phi_n(x) = \sum_{-N \leq j_k \leq M} \prod_{k=1}^{n} p_{j_k} \phi_0(\underbrace{2^n x - 2^{n-1} j_n - 2^{n-2} j_{n-1} - \dots - j_1}_{=2^n x - \sum_{m=0}^{n-1} 2^m j_{m+1}})$$

This implies that the function ϕ_0 is first dilated by a factor 2^n which reduces the support from $[0,1]$ to $[0, 1/2^n]$. The maximum translates of this dilated function to the left & right are determined then by $-N$ & M respectively, i.e. the maximum translation of the support to the left is

$$-\sum_{m=0}^{n-1}(-N)\,2^m = N\cdot\sum_{m=0}^{n-1} 2^m = N\cdot\frac{2^n-1}{2-1} = N(2^n-1)$$

similarly to the right we get $-M(2^n-1)$

so that the support of ϕ_n is given by

$$\left[0-\frac{N(2^n-1)}{2^n}\,,\ 1/2^n+\frac{M(2^n-1)}{2^n}\right]$$

$$=\left[-N+\frac{N}{2^n}\,,\ 1/2^n+M-M/2^n\right]$$

which upon taking limits as $n\to\infty$ gives for $\phi(x)$ (if it exists & the sequence of fcts converges)

$\operatorname{supp}(\phi)\subseteq[-N,M]$ clearly this is compact. $\square$

Chapter 5, problem 18:

We read off from our previous work above

$\operatorname{supp}(\phi)\subseteq[0,N]$ (which is compact) $\square$