Math 424 - Assignment 9 - Solutions

Chap. 6, problem 2:

$$\hat{\psi}_N^{(k)}(\xi) = \frac{d^{(k-1)}}{d\xi^{(k-1)}}\Big[\tfrac{1}{2} i e^{-i\xi/2} P_N(-e^{i\xi/2})\hat{\phi}_N(\xi/2) +$$

$$- e^{-i\xi/2} P_N'(-e^{i\xi/2})\cdot(-\tfrac{i}{2}e^{i\xi/2})\hat{\phi}_N(\xi/2)$$

$$+ (-e^{-i\xi/2})\cdot P_N(-e^{i\xi/2})\,\hat{\phi}_N'(\xi/2)\cdot \tfrac{1}{2}\Big]$$

$= \ldots$ etc.

Applying the product & chain rules $k-1$ more times we are going to obtain terms involving the derivatives of P_N for all orders up to k;

From (6.6.) p. 231 we know

$$P_N(z) = (z+1)^N \tilde{P}_N(z) \quad \& \quad \tilde{P}_N(-1) \neq 0$$

so $$P_N'(z) = N(z+1)^{N-1}\tilde{P}_N(z) + (z+1)^N \tilde{P}_N'(z)$$

$$P_N''(z) = N(N-1)(z+1)^{N-2}\tilde{P}_N(z) + 2N(z+1)^{N-1}\tilde{P}_N'(z) + (z+1)^N \tilde{P}_N''(z)$$

$\vdots$

etc.

if we evaluate at $z=-1$ all terms will vanish if they contain $(z+1)^k$ $k>0$ & we are left with

$$(*)\begin{cases} P_N^{(k)}(-1) = 0 & \text{if } k < N \\ P_N^{(N)}(-1) = N!\,\tilde{P}_N(-1) \end{cases}$$

Since we want to find $\hat{\psi}_N^{(k)}(0)$ and $e^{i\xi/2}|_{\xi=0} = 1$ we find that

$$\hat{\psi}_N^{(k)}(0) = 0 \quad \text{for } k < N \quad \text{(as every term contains } P_N^{(m)}(-1) \text{ for } m \leq k)$$

and

$$\hat{\psi}_N^{(N)}(0) = P_N^{(N)}(-e^{i0}) \cdot (-1) \cdot e^{i0} (i/2)^N \cdot (-e^{-i0}) \hat{\phi}_N(0)$$

$$= -N! \, \tilde{P}_N(-1) \, (i/2)^N \, 1/\sqrt{2\pi}$$

where we used that $\hat{\phi}_N(0) = 1/\sqrt{2\pi}$

since $\hat{\phi}_N(0) = 1/\sqrt{2\pi} \prod_{j=1}^{\infty} P_N(1) = 1/\sqrt{2\pi} \cdot \prod_{j=1}^{\infty} 1 = 1/\sqrt{2\pi}$

($P_N(1) = 1$, Thm 5.23.)

□

Chap. 6, problem 3:

$$\int_0^{3\cdot 2^{-j}} \left(f(2^{-j}k) + x\, f'(2^{-j}k) + \tfrac{1}{2}x^2 f''(2^{-j}k) \right) 2^{j/2}\, \psi_2(2^j x)\, dx$$

$$= \underbrace{\int_0^{3\cdot 2^{-j}} f(2^{-j}k)\, 2^{j/2}\, \psi_2(2^j x)\, dx}_{:=A} + \underbrace{\int_0^{3\cdot 2^{-j}} x\, f'(2^{-j}k)\, 2^{j/2}\, \psi_2(2^j x)\, dx}_{:=B}$$

$$+ \underbrace{\int_0^{3\cdot 2^{-j}} \frac{1}{2} x^2 f''(2^{-j}k)\, 2^{j/2}\, \psi_2(2^j x)\, dx}_{:=C}$$

$$A = \int_0^3 f(2^{-j}k)\, 2^{j/2}\, \psi_2(z)\, 2^{-j}\, dz$$

set $x = 2^{-j} z \Rightarrow \frac{dx}{dz} = 2^{-j} \Rightarrow dx = 2^{-j} dz$

$$= f(2^{-j}k)\, 2^{j/2} \int_0^3 \psi_2(z)\, dz = f(2^{-j}k)\, 2^{j/2} \int_{-\infty}^{\infty} \psi_2(z)\, dz$$

since supp $\psi_2 \subseteq [0,3]$

see remark p. 231

$= 0$ by using Proposition 6.1.

Similarly $B = 0$. For C we calculate

$$C = 2^{j/2-1} f''(2^{-j}k) \int_0^{3\cdot 2^{-j}} x^2\, \psi_2(2^j x)\, dx$$

$$= 2^{j/2-1} f''(2^{-j}k) \int_0^{3} 2^{-2j} z^2\, \psi_2(z)\, dz$$

$\text{supp}\, \psi_2 \subseteq [0,2]$

$$= 2^{-2j-1-j/2} f''(2^{-j}k) \int_{-\infty}^{\infty} z^2 \psi_2(z)\, dz$$

$$= 2^{-2j-1-j/2} f''(2^{-j}k) \left(-\tfrac{1}{8}\sqrt{\tfrac{3}{2\pi}}\right)$$

using (6.11) and so

$$= -\tfrac{1}{16}\sqrt{\tfrac{3}{2\pi}}\, f''(2^{-j}k)\, 2^{-5/2 j}$$

□

Chap. 6, problem 4:

(see my webpage at http://www.cam.cornell.edu/~ck274/ → Math424/); download all files into a directory and look & run Math424.m & Math424_1.m

NOTE: The code is by no means good or robust!