Smooth bumps, a Borel theorem and partitions of unity on p.c.f. fractals.*

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July 25, 2006

1 Introduction

Recent years have seen considerable developments in the theory of analysis on certain fractal sets from both probabilistic and analytic viewpoints [1, 10, 19]. In this theory, either a Dirichlet energy form or a diffusion on the fractal is used to construct a weak Laplacian with respect to an appropriate measure, and thereby to define smooth functions. As a result the Laplacian eigenfunctions are well understood, but we have little knowledge of other basic smooth functions except in the case where the fractal is the Sierpinski Gasket [15, 5, 16]. At the same time the existence of a rich collection of smooth functions is crucial to several aspects of classical analysis, where tools like smooth partitions of unity, test functions and mollifications are frequently used. In this work we give two proofs of the existence of smooth bump functions on fractals, one taking the probabilistic and the other the analytic approach. The probabilistic result (Theorem 2.1) is valid provided the fractal supports a heat operator with sub-Gaussian bounds, as is known to be the case for many interesting examples [1, 2, 3] that include non-post-critically finite (non-p.c.f.) fractals such as certain Sierpinski carpets. By contrast the analytic method (Theorem 3.8) is applicable to self-similar p.c.f. fractals with a regular harmonic structure and Dirichlet energy in the sense of Kigami [10].

For p.c.f. fractals we use our result on the existence of bump functions to prove a Borel-type theorem, showing that there are compactly supported smooth functions with prescribed jet at a junction point (Theorem 4.3). This gives a very general answer to a question raised in [15, 5], and previously solved only for the Sierpinski Gasket [16]. We remark, however, that even in this special case the results of [16] neither contain nor are contained in the theorem proven here, as the functions in [16] do not have compact support, while those here do not deal with the tangential derivatives at a junction point.

Finally we apply our Borel theorem to the problem of partitioning smooth functions. Multiplication does not generally preserve smoothness in the fractal setting [4],

*MSC Primary 28A80; Secondary 31C45, 60J60

†Research supported in part by the National Science Foundation, Grant DMS-0140194

so the usual partition of unity method is not available. As a substitute for this classical tool we show that a smooth function can be partitioned into smooth pieces with supports subordinate to a given open cover (Theorem 5.1).

Setting

Let X be a self-similar subset of $\mathbb{R}^d$ (or more generally any complete metric space) in the sense that there is a finite collection of contractive similarities $\{F_j\}_{j=1}^N$ of the space and X is the unique compact set satisfying $\cup_{j=1}^N F_j(X)$. The sets $F_j(X)$ are the 1-cells, and for a word $w = (w_1, w_2, \dots, w_m)$ of length m we define $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and call $F_w(X)$ an m-cell. If w is an infinite word then we define $[w]_m$ to be its length m truncation and let $F_w(X) = \cap_m F_{[w]_m}(X)$, which is clearly a point in X. The map from infinite words to X is surjective but not injective, and the points of non-injectivity play an important role in understanding the connectivity properties of the fractal (see Section 1.6 of [10]). In particular there are critical points of the cover by the F_j, namely those x and y for which there are $j \neq k$ in $\{1, \dots, N\}$ such that $F_j(x) = F_k(y)$ (so $F_j(x)$ is a critical value). We call an infinite word w critical if $F_w(X)$ is a critical value, and then call $\tilde{w}$ post-critical if there is $j \in \{1, \dots, N\}$ such that jw is critical. The boundary ∂X of X consists of all points $F_{\tilde{w}}(X)$ for which $\tilde{w}$ is post-critical. In the case that the set of post-critical words is finite the fractal is called post-critically finite (p.c.f.) and we also use the notations $V_0 = \partial X$ and $V_m = \cup_w F_w(V_0)$, where the union is over all words of length m. The points in $(\cup_m V_m) \setminus V_0$ are called *junction points*. We shall always assume that V_0 contains at least two elements.

We suppose that X comes equipped with a self-similar probability measure μ, meaning that there are $\mu_1, \dots, \mu_N$ such that the cell corresponding to $w = (w_1, \dots, w_m)$ has measure $\mu(F_w(X)) = \prod_{j=1}^m \mu_{w_j}$. In order to do analysis on X we assume that X admits a Dirichlet form $\mathcal{E}$, so $\mathcal{E}$ is a closed quadratic form on $L^2(\mu)$ with the (Markov) property that if $u \in \mathrm{dom}(\mathcal{E})$ then so is $\tilde{u} = u\chi_{0<u<1} + \chi_{u\geq 1}$ and $\mathcal{E}(\tilde{u}, \tilde{u}) \leq \mathcal{E}(u, u)$, where χ_A is the characteristic function of A. We will work only with self-similar Dirichlet forms, having the property that

$$\mathcal{E}(u, v) = \sum_{m\text{-words w}} r_w^{-1} \mathcal{E}(u \circ F_w, v \circ F_w) \tag{1.1}$$

where the factors r_j are called resistance renormalization factors and as usual $r_w = r_{w_1} \cdots r_{w_m}$. For convenience we restrict to the case of regular harmonic structures, in which $0 < r_j < 1$ for all j. In addition we assume $\mathcal{E}$ has the property that $C(X) \cap \mathrm{dom}(\mathcal{E})$ is dense both in $\mathrm{dom}(\mathcal{E})$ with $\mathcal{E}$-norm and in the space of continuous functions $C(X)$ with supremum norm. We often refer to $\mathcal{E}$ as the energy. If X is a nested fractal in the sense of Lindstrøm [13] then such a Dirichlet form may be constructed using a diffusion or a harmonic structure [12, 7, 18]. Other approaches may be found in [17, 11, 14, 9].

Using the energy and measure we produce a weak Laplacian by defining $f = \Delta u$ if $\mathcal{E}(u, v) = -\int f v \, d\mu$ for all $v \in \mathrm{dom}(\mathcal{E})$ that vanish on ∂X. Then $-\Delta$ is a non-negative self-adjoint operator on $L^2(\mu)$. When $\Delta u \in C(X)$ we write $u \in \mathrm{dom}(\Delta)$; this notation is continued inductively to define $\mathrm{dom}(\Delta^k)$ for each k and then $\mathrm{dom}(\Delta^\infty) = \cap_k \mathrm{dom}(\Delta^k)$. We say f is smooth if $f \in \mathrm{dom}(\Delta^\infty)$. On a p.c.f. fractal the weak Laplacian admits an additional pointwise description in which $\mathcal{E}$ is a renormalized limit of energies $\mathcal{E}_m$ corresponding to the finite graph approximations V_m and the Laplacian Δ is a renormalized limit of the associated graph Laplacians Δ_m. Details are in Section 3.7 of [10].

By standard results, existence of the Dirichlet form $\mathcal{E}$ implies existence of a strongly continuous semigroup $\{P_t\}$ with generator $-\Delta$. Conversely if there is such a semigroup and it is self-adjoint then there is a corresponding Dirichlet form, so we could equally well begin with $\{P_t\}$ and construct $\mathcal{E}$ (see [6, 8]). The Markov property of $\mathcal{E}$ ensures that if $0 \leq u \leq 1$ μ-a.e. then also $0 \leq P_t u \leq 1$ μ-a.e.

If X is p.c.f. then there is a definition of boundary normal derivatives of a function in $\mathrm{dom}(\Delta)$ and a Gauss-Green formula relating these to the integral of the Laplacian on X. The usual definition uses resistance-renormalized limits of the terms of the graph Laplacian that exist at the boundary point. If q_i is the boundary point of X that is fixed by F_i and r_i is the resistance factor corresponding to F_i we may define a normal derivative ∂_n at q_i and have a Gauss-Green formula as in Section 3.7 of [10] by

$$\partial_n u(q_i) = -\lim_{m\to\infty} r_i^{-m} \Delta_{m,q_i} u(q_i) \tag{1.2}$$

$$\sum_{q\in\partial X} (v(q)\partial_n u(q) - u(q)\partial_n v(q)) = \int_X (v\Delta u - u\Delta v)\, d\mu \tag{1.3}$$

where in (1.2) the quantity $\Delta_{m,q_i} u(q_i)$ is the usual graph Laplacian with the terms that are not defined at q_i omitted. Normal derivatives may also be localized to cells, so that $\partial_{n,F_w(X)} u(F_w(q_i))$ is given by the limit in (1.2) but with $\Delta_{m,w} u(F_w(q_i))$ omitting terms that involve points outside $F_w(X)$. It is then easy to see that if Δu exists and is continuous on each of finitely many cells that meet at $F_w(q_i)$, then it is continuous on their union if and only if the following conditions hold: u is continuous, Δu has a unique limit at $F_w(q_i)$, and the normal derivatives at $F_w(q_i)$ sum to zero. We call these the *matching conditions* for the Laplacian.

We shall have need of two other pieces of information about a p.c.f. fractal with regular harmonic structure. The first is that there is a Green's function $g(x, y) \geq 0$ that is continuous on $X \times X$ and has self-similar structure related to the discrete Greens function $\Psi(x, y)$ on $(V_1 \setminus V_0)^2$. According to Section 3.5 of [10]

$$g(x, y) = \lim_{m\to\infty} \sum_{k=0}^{m-1} \sum_{w\in W_k} r_w \Psi(F_w^{-1}(x), F_w^{-1}(y)) \tag{1.4}$$

where W_k is the collection of words of length k but the sum is only over those w such that $F_w^{-1}(x)$ and $F_w^{-1}(y)$ make sense. Integration against (the negative of) this Green's function gives the Green's operator $Gf(x) = -\int g(x, y) f(y) d\mu(y)$ and solves $\Delta G f = f$ with Dirichlet boundary conditions. In particular we will make use of pointwise estimates of $g(x, y)$ that follow from (1.4). The second thing we need to know is an estimate on the oscillation of a harmonic function on a cell $F_w(X)$, details of which are in Section 3.2 and Appendix A of [10]. A harmonic function $h(x)$ is determined by its values on the boundary V_0, and its values may be computed using harmonic extension matrices A_i via $h|_{F_w(V_0)} = A_{w_m} \cdots A_{w_1} \cdot h|_{V_0}$. The A_i are positive definite, have eigenvalue 1 on the constant function and second eigenvalue at most r_i. It follows immediately that the oscillation of $A_i h|V_0$ is at most r_i when the oscillation of $h|_{V_0}$ is bounded by 1, and similarly that the oscillation of $h|_{F_w(X)}$ is at most r_w.

More details about analysis on self-similar p.c.f. fractals may be found in [10], or [19] in the special case of the Sierpinski Gasket. The lecture notes of Barlow [1] cover the probabilistic approach that begins with a diffusion semigroup; non-p.c.f. examples include Sierpinski carpets [2, 3]. Some of the general theory connecting Dirichlet forms, heat semigroups and spectral theory of the Laplacian is covered in [6, 8].

Smooth bump functions

In classical analysis on Euclidean spaces the usual bump functions to consider are of the form $u \in C^\infty$ with support in a specified open set Ω, bounds $0 \leq u \leq 1$ and the property $u \equiv 1$ on a specified compact $K \subset \Omega$. In the fractal setting described above it is not usually the case that a product of smooth functions is itself smooth (see [4], or Section 5 below), so there is less practical benefit to asking that our bump functions be identically 1 on K and we will not always do so. Nor is it always essential that $0 \leq u \leq 1$, though this is sometimes useful. For this reason we will use the term *smooth bump function* to mean a function $u \in \operatorname{dom}(\Delta^\infty)$ with support in a specified set Ω and a bound $|u - 1| \leq \epsilon$ on a specified compact $K \subset \Omega$.

Suppose X is a p.c.f. self-similar fractal and we have a function $u \in \operatorname{dom}(\Delta^\infty)$ with $|u - 1| \leq \epsilon$ on $K \subset X$ and such that $\Delta^k u$ and $\partial_n \Delta^k u$ vanish on V_0 for all k. Then for any word w we see that

$$u_w = \begin{cases} u \circ F_w^{-1} & \text{on the cell } F_w(X) \\ 0 & \text{elsewhere} \end{cases}$$

is a smooth bump function with support in $F_w(X)$ by the matching conditions for the Laplacian. For this reason we also use the term *smooth bump function* to refer to $u \in \operatorname{dom}(\Delta^\infty)$ on X with $|u - 1| \leq \epsilon$ on K and which *vanishes to infinite order* at all points $q \in V_0$, by which last phrase we mean $\Delta^k u(q) = \partial_n \Delta^k u(q) = 0$ for all k.

2 A smooth bump from the heat operator

In this section (X, dist) is a metric space with measure μ. We require that there be a self-adjoint Neumann Laplacian Δ, and we make two assumptions on Δ. The first is that it has a positive spectral gap, in the sense that there is $\lambda > 0$ such that the spectrum of Δ is contained in $\{0\} \cup [\lambda, \infty)$. This implies the estimate $\|P_t - I\|_{2,2} \leq \min\{\lambda t, 2\}$, where $\|\cdot\|_{2,2}$ refers to the operator norm on L^2, P_t is the heat operator at time t and I is the identity. Secondly we assume that the diffusion Y_t associated to Δ satisfies the sub-Gaussian bound

$$\sup_x \mathbb{P}^x\{\operatorname{dist}(Y_t, x) \geq d\} \leq \frac{\gamma_1}{t^{\alpha/\beta}} \exp\left(-\gamma_2\Big(\frac{d^\beta}{t}\Big)^{1/(\beta-1)}\right). \tag{2.1}$$

A stronger assumption would be that P_t has a transition density $p(t, x, y)$ such that (2.1) remains true when the left side is replaced by $p(t, x, y)$ for any x and y with $\operatorname{dist}(x, y) < d$. Heat kernel bounds of this type on fractals have been the subject of a great deal of research; here we satisfy ourselves with noting that they are known for many examples, including various p.c.f. fractals [12, 7, 18] and certain highly symmetric generalized Sierpinski carpets (which are not p.c.f.) [2, 3]. They are also valid on products of some fractals [20].

For convenience we also assume that our space has finite measure M, though it will be clear in the proof that a weaker assumption would suffice. For example if we consider only compact K in Theorem 2.1 it would be enough to know that X is locally compact and μ is Radon.

Theorem 2.1. *For any K and $\epsilon > 0$ there is a nonnegative infinitely smooth function v such that $v \equiv 1$ in K, and $v \equiv 0$ in K_ϵ^c, where K_ϵ is the ϵ-neighborhood of K, and K_ϵ^c is its complement.*

When reading the proof it may be helpful to have the following heuristic in mind. Our goal is a function with two properties, the first of which is smoothness and the second is the property of being $\equiv 1$ on K and $\equiv 0$ on K_ϵ^c, which we call the *characteristic* property. Beginning with a characteristic function $u_0 = \chi_{A_0}$ where $K \subset A_0 \subset K_\epsilon$, which has the characteristic property but is not smooth, we recursively apply a two step method. The first step smoothes u_0 by applying the heat operator for a small time t_1 to obtain $v_1 = P_{t_1} u_0$, which is smooth but does not have the characteristic property. The second step replaces v_1 by a constant a_1 on a neighborhood A_1 of K and another constant b_1 on a neighborhood B_1 of K_ϵ^c, then translates and rescales as stated in (2.6) to obtain a function u_1 which has the characteristic property but is again non-smooth. What we have gained in passing from u_0 to u_1 is replacing the original abrupt drop of the characteristic function with the improved piece between A_1 and B_1, as illustrated on Figure 1. This argument is repeated inductively, each time applying the heat kernel for a shorter time t_j to get $v_j = P_{t_j} u_{j-1}$ and then cutting v_j off above at height a_j on a neighborhood A_j with $K \subset A_j \subset A_{j-1}$ and below at height b_j on B_j with $K_\epsilon^c \subset B_j \subset B_{j-1}$ to get u_j. Once we know that $a_j \to 1$ and $b_j \to 0$ it is unsurprising that this process converges (say in L^2). What is perhaps unexpected is that the result is smooth, and this is where the sub-Gaussian estimate (2.1) is crucial. Essentially what is going on is that the "steepness" of the $(j+1)$-th interpolant depends both on the height it must interpolate and the "width scale" on which it interpolates. The "width scale" depends on t_{j+1} through the norm $\|\Delta^k P_{t_{j+1}}\|_{2,2} \lesssim t_{j+1}^{-k}$ for the "steepness" measured by Δ^k, but the height to be interpolated depends instead on how much P_{t_j} changed the function during the smoothing step, and this quantity is exponentially small in t_j by (2.1). The fact that the "steepness" measured by Δ^k is of size $t_{j+1}^{-k}(a_j + b_j)$ is shown in Lemma 2.5, while Lemma 2.2 is where we exploit the sub-Gaussian estimate (2.1) to choose times $\{t_j\}$ such that $(a_j + b_j)t_{j+1}^{-k}$ is summable. Lemmas 2.3 and 2.4 are the necessary substeps showing L^2 convergence; as expected they do not require the full strength of the conclusions of Lemma 2.2.

Proof. Let $A_j = K_{\epsilon 2^{-(j+j_0)}}$ and $B_j = K^c_{\epsilon(1-2^{-(j+j_0)})}$, where j_0 will be chosen later. Induc-

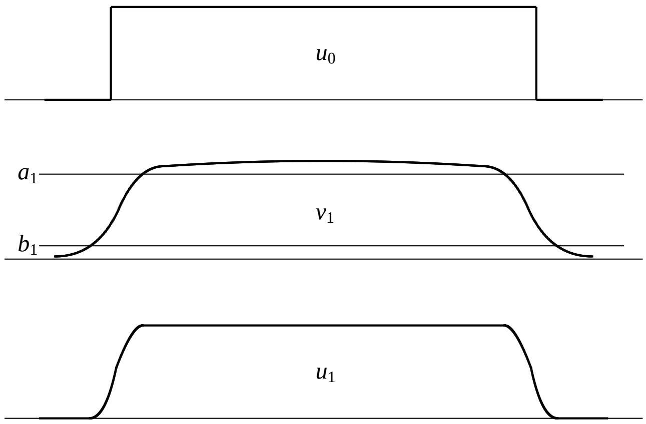

Figure 1: Initial steps of the bump construction

tively define smoothed functions v_j and cut-off versions u_j of the v_j as follows.

$$u_0 = \chi_{A_0} \tag{2.2}$$
$$v_j = P_{t_j} u_{j-1} \tag{2.3}$$
$$a_j = 1 - \inf_{A_j} u_{j-1} \tag{2.4}$$
$$b_j = \sup_{B_j} v_j \tag{2.5}$$
$$u_j(x) = \begin{cases} 1 & \text{if } v_j(x) \geq 1 - a_j \\ \frac{v_j(x) - b_j}{1 - b_j - a_j} & b_j < v_j(x) < 1 - a_j \\ 0 & \text{if } v_j(x) \leq b_j \end{cases} \tag{2.6}$$

where χ_{A_0} is the characteristic function of A_0. We see from Lemma 2.2 that a good choice of j_0 ensures $a_j + b_j \leq \frac{1}{2}$ so that (2.6) is well defined. It is clear that $0 \leq u_j \leq 1$ with $u_j \equiv 1$ on A_j and $u_j \equiv 0$ on B_j. Moreover Lemma 2.4 shows u_j and v_j converge to the same limit v in L^2, and from Lemma 2.5 we see that for each $k \geq 0$ the sequence $\{\Delta^k v_j\}$ is L^2 Cauchy. It follows that v is smooth and non-negative, and that $v \equiv 1$ on K and $v \equiv 0$ on K_ϵ^c, so the theorem follows. □

Lemma 2.2. *There is j_0 and a decreasing sequence of times $t_j \leq \frac{2}{\lambda}$ with the property that $a_j + b_j \leq \frac{1}{2}$ for all $j \geq 1$, that $\sum_1^\infty t_j = T < \infty$, and for all $k \geq 0$ we have*

$$\sum_1^\infty t_{j+1}^{-k}(a_j + b_j) = C(k) < \infty.$$

Proof. Observe that if $x \in A_j$ then $u_{j-1} \equiv 1$ in a ball of radius $\epsilon 2^{-(j+j_0)}$ around x. Similarly if $y \in B_j$ then $u_{j-1} \equiv 0$ in a ball of radius $\epsilon 2^{-(j+j_0)}$ around y. It follows from this, the assumption (2.1), and $0 \le u_{j-1} \le 1$ that

$$\begin{aligned}
a_j &\le \sup_{x\in A_j} |1 - v_j(x)| \\
&\le \sup_{x\in A_j} \mathbb{P}^x\{\operatorname{dist}(Y_{t_j}, x) > \epsilon 2^{-(j+j_0)}\} \\
&\le \frac{\gamma_1}{t_j^{\alpha/\beta}} \exp\Big(-\gamma_2\Big(\frac{\epsilon 2^{-(j+j_0)}}{t_j}\Big)^{1/(\beta-1)}\Big)
\end{aligned}$$

and the same bound is valid for b_j by the same method. It is then easy to find superexponentially decaying t_j that have the required properties, for example $t_j = (j+j_0)^{-(j+j_0)}$, or $t_j = 2^{-(j+j_0)^2}$ both work once we choose j_0 so large that $t_j \le \frac{2}{\lambda}$ and $a_j + b_j \le \frac{1}{2}$ for $j \ge 1$. ◻

Lemma 2.3. $\|v_j\|_2 \le 2^{2C(0)} M^{1/2}$.

Proof. For $t \ge 0$ the heat operator P_t is an L^2 contraction, so $\|v_{j+1}\|_2 = \|P_{t_{j+1}} u_j\|_2 \le \|u_j\|_2$. Recognizing that u_j can be written as

$$u_j = \frac{1}{1 - a_j - b_j} \max\{0, \min\{(1 - a_j), v_j\} - b_j\}$$

we see immediately that $\|u_j\|_2 \le (1 - a_j - b_j)^{-1} \|v_j\|$, whence

$$\|v_{j+1}\|_2 \le \|u_0\|_2 \prod_{l=1}^{j} (1 - a_l - b_l)^{-1} \le M^{1/2} \exp\Big(2 \ln 2 \sum_1^\infty (a_l + b_l)\Big) = 2^{2C(0)} M^{1/2}$$

◻

Lemma 2.4. $\|v_j - u_j\|_2 \le 3M^{1/2}(a_j + b_j)$.

Proof. Compute

$$\begin{aligned}
\|v_j - u_j\|_2^2 &= \int_{v_j \le b_j} |v_j|^2 + \int_{v_j \ge 1 - a_j} |1 - v_j|^2 + \int_{b_j \le v_j \le 1 - a_j} \Big| \frac{v_j - b_j}{1 - a_j - b_j} - v_j \Big|^2 \\
&\le M b_j^2 + M a_j^2 + \frac{1}{(1 - a_j - b_j)^2} \int \Big((a_j + b_j) v_j - b_j\Big)^2 \\
&\le M b_j^2 + M a_j^2 + \frac{M (a_j + b_j)^2}{(1 - a_j - b_j)^2} \\
&\le 5M (a_j + b_j)^2
\end{aligned}$$

◻

Lemma 2.5. *For each $k \ge 0$ the sequence $\{\Delta^k v_j\}$ is L^2-Cauchy.*

Proof. Using that the heat operator commutes with Δ, as well as the estimate $\|\Delta^k P_t\|_{2,2} \leq c_k t^{-k}$ from the spectral representation $P_t = \int_0^\infty e^{-xt} dE_\Delta(x)$ we find

$$\begin{aligned}
\left\|\Delta^k(v_{j+1} - v_j)\right\|_2 &= \left\|\Delta^k(P_{t_{j+1}} u_j - v_j)\right\|_2 \\
&\leq \left\|\Delta^k P_{t_{j+1}}(u_j - v_j)\right\|_2 + \left\|\Delta^k(P_{t_{j+1}} v_j - v_j)\right\|_2 \\
&\leq \left\|\Delta^k P_{t_{j+1}}\right\|_{2,2} \left\|u_j - v_j\right\|_2 + \left\|(P_{t_{j+1}} - I)\Delta^k v_j\right\|_2 \\
&\leq 3M^{1/2} c_k t_{j+1}^{-k}(a_j + b_j) + \lambda t_{j+1} \|\Delta^k v_j\|_2
\end{aligned} \tag{2.7}$$

where in the final step we used the bound from Lemma 2.4, the spectral gap assumption that $\|P_t - I\|_{2,2} \leq \min \lambda t, 2$ and the fact that all $t_j \leq \frac{2}{\lambda}$. Then

$$\|\Delta^k v_{j+1}\|_2 \leq 3M^{1/2} c_k t_{j+1}^{-k}(a_j + b_j) + (1 + \lambda t_{j+1})\|\Delta^k v_j\|_2$$

and by induction and Lemma 2.2

$$\|\Delta^k v_{j+1}\|_2 \leq 3M^{1/2} c_k \sum_{l=0}^{\infty} \Big(t_{l+1}^{-k}(a_l + b_l) \prod_{m=l}^{\infty} (1 + \lambda t_{j+1}) \Big) \leq 3M^{1/2} c_k C(k) e^{\lambda T}$$

Upon substitution into (2.7) we find

$$\left\|\Delta^k(v_{j+1} - v_j)\right\|_2 \leq 3M^{1/2} c_k t_{j+1}^{-k}(a_j + b_j) + 3M^{1/2} c_k C(k) e^{\lambda T} \lambda t_{j+1}$$

which is summable by Lemma 2.2. □

3 A smooth bump as a fixed point of an operator

To understand why it is sometimes possible to construct a smooth bump function on a self-similar set as a fixed point of an operator, we invite the reader to consider an elementary situation. Let $I = [0, 1]$ be the unit interval in $\mathbb{R}$. We may view I as a p.c.f. self-similar set under the contractions $f_0 = x/2$ and $f_1 = (x + 1)/2$. If μ is Lebesgue measure and $\mathcal{E}$ is defined using a limit of a regular self-similar harmonic structures with resistance factors $1/2$ then we obtain the usual Dirichlet energy and Laplacian, and the normal derivatives are the outward-directed one-sided derivatives at 0 and 1 (see [10, 19] for details).

The intuition for our construction is as follows. Consider a symmetric smooth bump function u on the interval $I = [0, 1]$, for which $u \equiv 1$ on $[L, 1-L]$ as shown in Figure 2. If we look at the graph of $\Delta u = d^2u/dx^2$ we obtain something that looks like a constant multiple of Figure 3, which appears as if it could be assembled from rescaled copies of u according to a rule like

$$\Phi u = \begin{cases} u\left(\frac{2x}{L}\right) & \text{if } 0 \leq x \leq \frac{L}{2} \\ -u\left(\frac{2x}{L} - 1\right) & \text{if } \frac{L}{2} < x \leq L \\ 0 & \text{if } L < x < 1 - L \\ -u\left(\frac{2x-2}{L} + 2\right) & \text{if } 1 - L \leq x < 1 - \frac{L}{2} \\ u\left(\frac{2x-2}{L} + 1\right) & \text{if } 1 - \frac{L}{2} \leq x \leq 1 \end{cases} \tag{3.1}$$

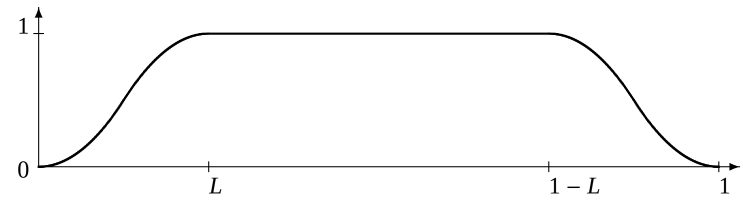

Figure 2: The smooth bump function u.

so that we might hope there is actually a smooth bump function u which has precisely this scaling behavior. If we let G denote the Green's operator for the operator Δ on I with Dirichlet boundary conditions, then this would be equivalent to asking that u be a fixed point of the operator

$$\Psi u(x) = \frac{G \circ \Phi u(x)}{G \circ \Phi u(1/2)} \tag{3.2}$$

It is a consequence of our general result Theorem 3.8 that the operator Ψ in (3.2) has

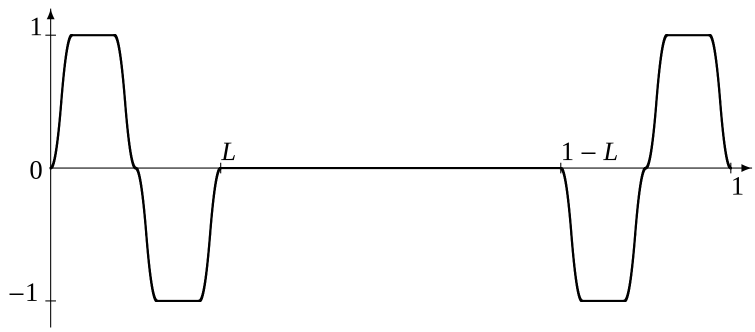

Figure 3: The function $\Delta u = d^2u/dx^2 = \Phi(u)$.

a fixed point and that the fixed point is a smooth bump function. In fact more is true in the special case of I, where the fact that removing any interior point disconnects the set, along with the existence of an explicit formula for the Green's function, allows us to prove that the fixed point has values in $[0, 1]$ and is identically 1 on $[L, 1 - L]$. For reasons of brevity we do not include the proof of this result; it is a simpler version of the proof of Theorem 3.8.

Proposition 3.1. *If L is sufficiently small then the operator Ψ preserves the space of continuous functions on I that have values in $[0, 1]$, vanish at 0 and 1, and are identically 1 on $[L, 1 - L]$. Furthermore Ψ is a contraction in the L^∞ norm on these functions, and its fixed point is a smooth function that vanishes to infinite order at 0 and 1.*

Another example in which we can define operators Φ and Ψ that are similar to (3.1) and (3.2) is the Sierpinski gasket SG with its standard harmonic structure and measure, where for sufficiently large l we can set

$$\Phi u(x) = \begin{cases} 2u(F_i^{-(l+1)}(x)) & \text{if } x \in F_i^{(l+1)}(SG) \\ -u(F_j^{-1} \circ F_i^{-l}(x)) & \text{if } x \in F_i^l \circ F_j(SG),\ j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

and with p any vertex from V_1 let

$$\Psi u = \frac{G \circ \Phi u(x)}{G \circ \Phi u(p)} \tag{3.4}$$

as illustrated in Figure 4 for the case $l = 2$. Again we omit the variant of the proof of

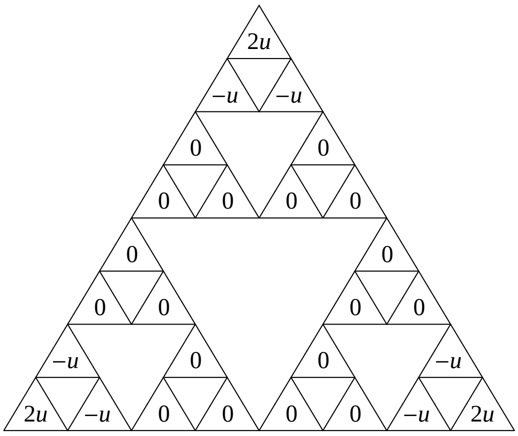

Figure 4: Φu in the case $l = 2$.

Theorem 3.8 that establishes the following result

Proposition 3.2. *The operator* Ψ *of* (3.4) *is an* L^∞*-contractive self-map of the set of functions that are continuous on* SG*, vanish at the boundary, are identically* 1 *on* $SG \setminus \cup_i F_i^l$*, and satisfy* $\left|\int u - 1\right| \leq \frac{1}{2}$*. The fixed point of* Ψ *is a smooth bump function.*

The method described for I and SG rely heavily on the symmetry of these sets and on the assumption that they are endowed with the symmetrical harmonic structures and measure. This assumption is unavoidable if we want to use the same operation Φ at all steps of the computation, as the natural linear combination of rescaled copies of the function will not otherwise have the desired properties, but it is very restrictive. Even some of the simplest of the nested fractals defined by Lindstrøm [13] have insufficient symmetry for a fixed Φ to be used in the construction of a smooth bump by this method. Nonetheless the method can be adapted to general p.c.f. self-similar fractals with regular harmonic structure and self-similar measure.

Let X be p.c.f. self-similar with boundary $V_0 = \partial X$, measure μ that is self-similar with scaling factors μ_j and regular harmonic structure with factors r_j. We fix a scale

l_1 with size to be determined later, and label the boundary l_1 cells by $Y_j = F_j^{l_1}(X)$. Their union is $Y = \cup Y_j$. For any $\epsilon > 0$ we will build a smooth function that satisfies $|u - 1| \leq \epsilon$ on $X \setminus Y$ by a construction that inductively determines its Laplacian on the cells Y_j, writing it as a fixed point of an operator Ψ on the following space of functions.

Definition 3.3. *Let C be the space of continuous functions u on X such that $u(q) = 0$ for $q \in \partial X$ and $\|u - 1\|_1 \leq \frac{1}{2}$. Note that this space is non-empty and closed in the continuous functions with supremum norm.*

To define the operator Ψ we need a little more notation. Let $S \subset V_{l_1}$ consist of those points that lie in some Y_j and in at least one other l_1-cell. If l_1 is sufficiently large then no two of the Y_j can intersect; we assume this and see that the connected components of $X \setminus S$ are the cells Y_j (less points of S) and the set $X \setminus Y$. Label those boundary points of the cell Y_j at which Y_j intersects another l_1 cell by $x_{i,j}$ for $i = 1, \dots, I_j$. Fixing a second scale l_2, also with size to be determined, we associate to each $x_{i,j}$ the unique $(l_1 + l_2)$-cell in Y_j containing $x_{i,j}$, calling it $Z_{i,j}$. We also set $Z_{0,j} = F_j^{l_1+l_2}(X)$, so it is the $(l_1 + l_2)$-cell in Y_j that contains $q_j \in V_0$, and define $w_{i,j}$ to be the word such that $F_{i,j}(X) = F_{w_{i,j}}(X) = Z_{i,j}$. Figure 5 illustrates our labelling conventions in the case $X = SG$, $l_1 = 2$ and $l_2 = 1$. We identify a particular function that is in C when l_1 is

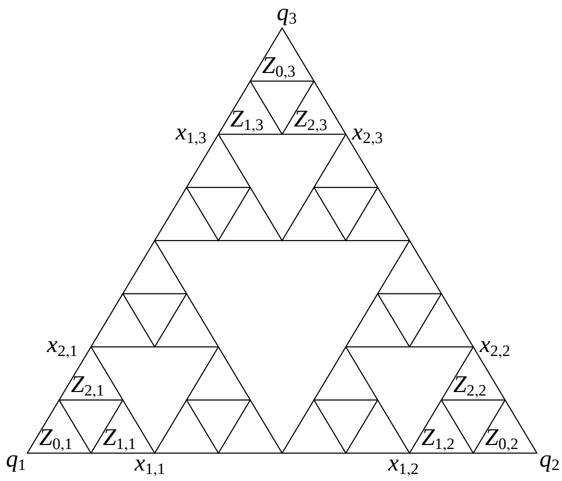

Figure 5: Notation if $X = SG$, $l_1 = 2$ and $l_2 = 1$.

large enough. Let f be the piecewise harmonic function on X with values $f(x_{i,j}) = 1$ for $i = 1, \dots, I_j$ and $j = 1, \dots, N$ but $f(x_{0,j}) = 0$ for all j. It is clear that f is continuous, identically 1 on $X \setminus Y$ and harmonic on each of the sets Y_j. It fails to be harmonic only at the points $x_{i,j}$ with $i \geq 1$, and we readily compute that the Laplacian of f is a measure supported at these points. In fact if δ_x denotes the Dirac mass at x then

$$\Delta f = \sum_{j=1}^{N} \sum_{i=1}^{I_j} a_{i,j} \delta_{x_{i,j}} = -\sum_{j=1}^{N} \sum_{i=1}^{I_j} \partial_n f_j(x_{i,j}) \delta_{x_{i,j}} \tag{3.5}$$

from which we also find that there is a constant $C(r)$, depending on the harmonic structure but not on the scales or locations, so that $|a_{i,j}| \leq C(r) r_j^{-l_1}$.

The smooth bump function we seek will actually be a perturbation of f, constructed by iteratively replacing the Dirac masses in (3.5) by rescaled copies of the stage k bump, correcting for the boundary normal derivatives, and applying the Dirichlet Green's operator to obtain the stage $k+1$ bump. We will see that each stage gains one order of smoothness, so the limiting function will be in $\mathrm{dom}(\Delta^\infty)$. Our first step is to estimate the effect of a perturbation of the type described.

Lemma 3.4. *For each $j = 1, \ldots, N$ and $i = 1, \ldots, I_j$ let $\nu_{i,j}$ be a finite, signed Borel measure with support in $Z_{i,j}$. If we use the coefficients in* (3.5) *to define*

$$\nu = \sum_{j=1}^{N} \sum_{i=1}^{I_j} a_{i,j} \nu_{i,j}$$

and let $u = G(\nu)$ be the result of applying the Dirichlet Green's operator, then

$$|u(x)| \leq C(r) N^2 \sup_{i,j} \|\nu_{i,j}\| \quad \textit{for all } x \in X \tag{3.6}$$

where $\|\nu_{i,j}\|$ is the total variation of $\nu_{i,j}$. If in addition we have $\int \nu_{i,j} = 0$ for all i and j then

$$|u(x)| \leq C(r) N \Big(\sup_{i,j} \|\nu_{i,j}\|\Big) \Big(\sum_{k=1}^{N} r_k^{l_2}\Big) \quad \textit{for all } x \in X \setminus Y \tag{3.7}$$

Proof. Recall that G may be represented as integration against the continuous kernel $-g(x,y)$, with sign chosen so $g(x,y) \geq 0$. The estimates we desire follow from (1.4) and the fact that $|a_{i,j}| \leq C(r) r_j^{-l_1}$. The former ensures both that $|g(x,y)| \leq C(r) r_j^{l_1}$ on each Y_j and that the oscillation of $g(x,y)$ on $Z_{i,j}$ is at most $C(r) r_{w_{i,j}}$. We compute

$$\begin{aligned}
|u(x)| &= \left| \int_X g(x,y) d\nu(y) \right| \\
&\leq \sum_{j=1}^{N} \sum_{i=1}^{I_j} |a_{i,j}| \int_{Z_{i,j}} |g(x,y)| d\nu_{i,j}(y) \\
&\leq \sum_{j=1}^{N} \sum_{i=1}^{I_j} C(r) r_j^{-l_1} \int_{Z_{i,j}} C(r) r_j^{l_1} d|\nu_{i,j}(y)| \\
&\leq C(r) N^2 \sup_{i,j} \|\nu_{i,j}\|
\end{aligned}$$

which establishes the first inequality. To obtain the second we observe that $\Delta u = 0$ on $X \setminus Y$, so by the maximum principle we need only verify the inequality at the points $x_{i,j}$. Fix such a point $x_{i',j'}$, and use that each $\int d\nu_{i,j} = 0$ to subtract the appropriate constant from each integrand before estimating:

$$
\begin{aligned}
|u(x_{i',j'})| &= \Big|\int_X g(x_{i',j'},y)d\nu(y)\Big| \\
&= \Big|\sum_{j=1}^{N}\sum_{i=1}^{I_j} a_{i,j}\int_{Z_{i,j}} g(x_{i',j'},y)d\nu_{i,j}(y)\Big| \\
&= \Big|\sum_{j=1}^{N}\sum_{i=1}^{I_j} a_{i,j}\int_{Z_{i,j}} \Big(g(x_{i',j'},y)-g(x_{i',j'},x_{i,j})\Big)d\nu_{i,j}(y)\Big| \\
&\le \sum_{j=1}^{N}\sum_{i=1}^{I_j} C(r)r_j^{-l_1}\int_{Z_{i,j}} C(r)r_{w_{i,j}}d|\nu_{i,j}| \\
&\le C(r)\sum_{j=1}^{N}\sum_{i=1}^{I_j} r_{k(i,j)}^{l_2}\|\nu_{i,j}\|
\end{aligned}
$$

because $r_{w_{i,j}} = r_j^{l_1} r_k^{l_2}$ for some $k = k(i, j)$. Each $k(i, j)$ occurs at most once for a fixed j, so we obtain

$$
|u(x_{i',j'})| \le C(r)\Big(\sup_{i,j}\|\nu_{i,j}\|\Big)\Big(\sum_{j=1}^{N}\sum_{k=1}^{N} r_k^{l_2}\Big) \le C(r)N\Big(\sup_{i,j}\|\nu_{i,j}\|\Big)\Big(\sum_{k=1}^{N} r_k^{l_2}\Big)
$$

□

There is an analogous but simpler estimate for the effect of introducing a mass supported on one of the small cells $Z_{0,j}$ at the boundary. If $\nu_{0,j}$ is a finite, signed Borel measure with support in $Z_{0,j}$ we use (1.4) to see $|g(x,y)| \le C(r)r_j^{l_1+l_2}$ on $Z_{0,j}$ and therefore

$$
\Big|G(\nu_{0,j})\Big| \le \Big|\int_{Z_{0,j}} |g(x,y)|d|\nu_{0,j}(y)| \le C(r)r_j^{l_1+l_2}\|\nu_{0,j}\| \tag{3.8}
$$

The ideas discussed so far allow us to generalize the definition of the operator Ψ in (3.2) and (3.4). The idea is that to replace the Dirac mass terms on the boundary cells $Z_{i,j}$ in (3.5) with normalized rescaled copies of $u \in C$ and apply the Green's operator to obtain a function that is near 1 on $X \setminus Y$. By adding some terms on the cells $Z_{0,j}$ we can make the result have vanishing normal derivatives at the boundary without changing the value on $X \setminus Y$ very much. In consequence we will obtain an operator that smooths $u \in C$ to be in $C \cap \mathrm{dom}(\Delta)$ with vanishing normal derivatives and is near 1 on $X \setminus Y$. Iterating the operator will then produce a sequence of smoother and smoother bump functions.

Let

$$
u_{i,j}(x) = \begin{cases} \mu(Z_{i,j})^{-1}\Big(\int_X u d\mu\Big)^{-1}\Big(u \circ F_{i,j}^{-1}(x)\Big) & \text{if } x \in Z_{i,j} \\ 0 & \text{otherwise} \end{cases} \tag{3.9}
$$

so that each $u_{i,j}$ is continuous and has integral 1. Since $u \in C$ we also have that $\int |u_{i,j}| \le \Big(\int u\Big)^{-1}\|u\| \le 3$.

Definition 3.5. *The operator Ψ on C is $\Psi u = G(v)$, where*

$$v(x) = \sum_{j=1}^{N} \sum_{i=0}^{I_j} b_{i,j} r_j^{-l_1} u_{i,j}(x) \tag{3.10}$$

and G is the Dirichlet Green's operator. In this expression the coefficients for $i \geq 1$ are given by $b_{i,j} = r_j^{l_1} a_{i,j}$ with $a_{i,j}$ as in (3.5)*, but the $b_{0,j}$ are yet to be determined.*

Note that $|b_{i,j}| \leq C(r)$ when $i \geq 1$. It is immediate that $G(v) \equiv 0$ on ∂X and is continuous. Moreover $\Delta G(v) = v$ is a linear combination of continuous functions, so $\Psi u \in \mathrm{dom}(\Delta)$. The next lemma uses the Gauss-Green formula to reduce finding the correct $b_{0,j}$ to a problem in linear algebra.

Lemma 3.6. *If l_1 and l_2 are sufficiently large then there are values $b_{0,j}$ such that $\partial_n \Psi u \equiv 0$ on ∂X. The minimal sizes of l_1 and l_2 depend only on the harmonic structure of X and the number of vertices N in ∂X.*

Proof. Let h_j be the function that is harmonic on X, equal to 1 at q_j and 0 at all other points of ∂X. Using $\Delta G(v) = v$, $G(v) \equiv 0$ on ∂X, and the Gauss-Green formula

$$\begin{aligned} \int h_j(y) v(y) d\mu(y) &= \int h_j(y) \Delta G(v)(y) d\mu(y) \\ &= \sum_{q_k \in \partial X} h_j(q_k) (\partial_n G(v))(q_k) \\ &= (\partial_n G(v))(q_j) \end{aligned}$$

from which $\partial_n G(v) \equiv 0$ on ∂X is simply

$$0 = \int h_j(y) v(y) d\mu(y) = \sum_{i',j'} b_{i',j'} r_{j'}^{-l_1} \int h_j(y) u_{i',j'}(y) d\mu(y)$$

for all $j = 1, \ldots, N$. Moving the terms depending on the fixed values $b_{i',j'}$ for $i' \geq 1$ this may be reformulated as

$$\sum_{j'} b_{0,j'} r_{j'}^{-l_1} \int h_j(y) u_{0,j'}(y) d\mu(y) = -\sum_{j'=1}^{N} \sum_{i'=1}^{I_{j'}} b_{i',j'} r_{j'}^{-l_1} \int h_j(y) u_{i',j'}(y) d\mu(y) \tag{3.11}$$

which we recognize as a matrix equation $\sum_{j'} M(u)_{j',j}\, r_{j'}^{-l_1} b_{0,j'} = A(u)_j$ with

$$\big(M(ud\mu)\big)_{j',j} = \int h_j(y) u_{0,j'}(y) d\mu(y) \tag{3.12}$$

$$\big(A(ud\mu)\big)_j = -\sum_{j'=1}^{N} \sum_{i'=1}^{I_{j'}} b_{i',j'} r_{j'}^{-l_1} \int h_j(y) u_{i',j'}(y) d\mu(y) \tag{3.13}$$

It is clear that we need to know $M = M(ud\mu)$ is invertible, but rather than prove this directly we do so by proving a perturbation estimate similar to Lemma 3.4 that will be useful later. To this end consider replacing each of the measures $u_{0,j'}d\mu$ in (3.12) with a copy of a different probability measure $d\sigma$ scaled and translated to give $d\sigma_{i',j'}$ supported on $Z_{i',j'}$. We call the result $M(d\sigma)$. The difference of these measures has mass zero, so we can compute an estimate involving the total variation of the measures

$$\begin{aligned}\left|M(ud\mu - d\sigma)_{j',j}\right| &= \left|\int h_j(y)(u_{0,j'}(y)d\mu(y) - d\sigma_{0,j'}(y))\right| \\ &\leq \left|\int (h_j(y) - h_j(x_{0,j'}))\big(u_{0,j'}(y)d\mu(y) - d\sigma_{0,j'}(y)\big)\right. \\ &\leq C(r)r_{j'}^{l_1+l_2}\|ud\mu - d\sigma\| \end{aligned} \tag{3.14}$$

because h_j is harmonic and therefore varies by at most $C(r)r_{j'}^{l_1+l_2}$ on each $Z_{0,j'}$. In particular if the measures $d\sigma_{i',j'}$ are Dirac masses at the points $x_{0,j'}$ then $M(d\sigma)$ is simply the identity, so (3.14) implies

$$\left|(I - M)_{j',j}\right| \leq C(r)r_{j'}^{l_1+l_2}$$

from which M is invertible when $l_1 + l_2$ is large, with $\|I - M^{-1}\| \leq C(N,r)\sum_j r_j^{l_1+l_2}$.

A similar perturbation argument can be made for $A(d\mu - d\sigma)$, where $A(d\sigma)$ is obtained by replacing each $u_{i',j'}d\mu$ by $d\sigma_{i',j'}$ in (3.13). Estimating the integral terms and using the bound $|b_{i',j'}| \leq C(r)$ we obtain

$$\left|A(ud\mu - d\sigma)_j\right| \leq C(N,r)\Big(\sum_i r_i^{l_2}\Big)\|ud\mu - d\sigma\| \tag{3.15}$$

however this is not the most useful thing we can do here. Instead we recognize that the bounds $|h_j(y) - 1| \leq C(r)r_j^{l_1}$ on Y_j and $|h_j(y)| \leq r_{j'}^{l_1}$ on $Y_{j'}$ for $j' \neq j$ ensure

$$\left|A(ud\mu)_j + \sum_{i'=1}^{I_j} b_{i',j}r_j^{-l_1}\right| \leq C(N,r)$$

so that combining this with our bound on $I - M^{-1}$ we have

$$\left|b_{0,j} - \sum_{i'=1}^{I_j} b_{i',j}\right| \leq C(N,r)r_j^{l_1} \tag{3.16}$$

If we examine the function f in (3.5) it is clear that the normal derivative at each point $x_{0,j}$ is $\sum_{i=1}^{I_j} b_{i,j}$, so our choice of $b_{0,j}$ is a small perturbation of that which would be used to cancel the normal derivatives of f. We also remark that this shows all $|b_{0,j}| \leq C(N,r)$. $\square$

If l_1 and l_2 are large enough then the values $b_{0,j}$ from Lemma 3.6 may be used to complete Definition 3.5 for the operator Ψ. Some key properties of this operator are summarized in the following lemma.

Lemma 3.7. *If l_1 and l_2 are sufficiently large then $\Psi(u) \in \mathcal{C} \cap \operatorname{dom}(\Delta)$ and*

$$\left\|\Psi u\right\|_\infty \leq C_1 \tag{3.17}$$

$$\left|\Psi u(y) - 1\right| \leq C_2 \sum_{j=1}^{N} r_j^{l_2} \quad \textit{for all } y \in X \setminus Y \tag{3.18}$$

where C_1, C_2 and the minimal sizes of l_1 and l_2 are constants depending only on the harmonic structure of X, the measure μ and the number of vertices N in ∂X.

Proof. Since

$$\Psi(u) = G\Big(\sum_{j=1}^{N}\sum_{i=1}^{I_j} a_{i,j}u_{i,j} + \sum_{j=1}^{N} b_{0,j}r_j^{-l_1}u_{0,j}\Big)$$

we obtain (3.17) from (3.6) and (3.8), and the fact that $|b_{0,j}| \leq C(N,r)$ for all j. The estimate (3.18) is only a little more difficult. Using $f(x) = 1$ and (3.5) on the set $X \setminus Y$ we see that

$$\begin{aligned}
\left|\Psi u(x) - 1\right| &= \left|\Psi u(x) - f(x)\right| \\
&= \left|G\Big(\sum_{j=1}^{N}\sum_{i=0}^{I_j} b_{i,j}r_j^{-l_1}u_{i,j}\Big) - G\Big(\sum_{j=1}^{N}\sum_{i=1}^{I_j} a_{i,j}\delta_{x_{i,j}}\Big)\right| \\
&\leq \left|G\Big(\sum_{j=1}^{N}\sum_{i=1}^{I_j} a_{i,j}\big(u_{i,j}d\mu - \delta_{x_{i,j}}\big)\Big)\right| + \sum_{j=1}^{N}\left|b_{0,j}r_j^{-l_1}G(u_{0,j}d\mu)\right| \\
&\leq C(N,r)\sup_{i,j}\left\|u_{i,j}d\mu - \delta_{x_{i,j}}\right\|\Big(\sum_{k=1}^{N} r_k^{l_2}\Big) + C(N,r)\Big(\sum_{j=1}^{N} r_j^{l_2}\Big)\|u_{0,j}d\mu\| \\
&\leq C(N,r)\Big(\sum_{j=1}^{N} r_j^{l_2}\Big)
\end{aligned} \tag{3.19}$$

where the estimate for the $b_{0,j}$ terms came from (3.8) and that for the $a_{i,j}$ terms is from (3.7) because $\int u_{i,j}d\mu = 1 = \int \delta_{x_{i,j}}$ and both are supported on $Z_{i,j}$.

Finally we check that $\|\Psi u - 1\|_1 \leq \frac{1}{2}$. Using the results we have so far

$$\begin{aligned}
\|\Psi u - 1\|_1 &\leq \int_{X\setminus Y} C(N,r)\Big(\sum_{j=1}^{N} r_j^{l_2}\Big)d\mu + \int_Y (1 + C_1)d\mu \\
&\leq C(N,r)\Big(\sum_{j=1}^{N} r_j^{l_2}\Big) + (1 + C_1)\mu(Y)
\end{aligned} \tag{3.20}$$

so that we can be sure $\Psi u \in \mathcal{C}$ if both l_1 and l_2 are sufficiently large, because $\mu(Y) \to 0$ as $l_1 \to \infty$. It has already been observed that $u \in \operatorname{dom}(\Delta)$ and $u \equiv 0$ on ∂X, so the lemma is proven. □

Finally we come to the main result of this section. The following theorem implements the idea that motivated our definition of Ψ, namely that Ψ smoothes functions in $\mathcal{C}$ and therefore its recursive application gives a bump function in $\operatorname{dom}(\Delta^\infty)$.

Theorem 3.8. *Given $\epsilon > 0$ there are l_1 and l_2 sufficiently large that Ψ has a fixed point u_0 in C with $|u - 1| \le \epsilon$ on $X \setminus Y$. The fixed point is a smooth bump function and every $u \in C$ has $\|\Psi^k u - u_0\|_\infty \to 0$ as $k \to \infty$.*

Proof. Let $u, \tilde{u} \in C$. We calculate $\Psi u - \Psi \tilde{u} = G(v - \tilde{v})$, where v and $\tilde{v}$ are as in (3.10). Beginning with a variant of the computation (3.19) we have

$$\begin{aligned}\left|G(v-\tilde{v})(x)\right| &\le \left|G\Big(\sum_{j=1}^{N}\sum_{i=1}^{I_j} a_{i,j}(u_{i,j}-\tilde{u}_{i,j})\Big)\right| + \left|G\Big(\sum_{j=1}^{N} r_j^{-l_1}(b_{0,j}u_{0,j}-\tilde{b}_{0,j}\tilde{u}_{0,j})\Big)\right| \\ &\le \left|G\Big(\sum_{j=1}^{N}\sum_{i=1}^{I_j} a_{i,j}(u_{i,j}-\tilde{u}_{i,j})\Big)\right| + \sum_{j=1}^{N} r_j^{-l_1}|\tilde{b}_{0,j}|\left|G(u_{0,j}-\tilde{u}_{0,j})\right| \\ &\quad + \sum_{j=1}^{N} r_j^{-l_1}\left|b_{0,j}-\tilde{b}_{0,j}\right||G(u_{0,j})(x)| \end{aligned}\tag{3.21}$$

which suggests we will need to know estimates for both $(u_{i,j} - \tilde{u}_{i,j})$ and $|b_{0,j} - \tilde{b}_{0,j}|$. Conveniently we can reduce the latter to the former using (3.14) and (3.15), because $b_{0,j}$ and $\tilde{b}_{0,j}$ are computed from equations of the form $\sum_j M(ud\mu)_{j,i} b_{0,j} r_j^{-l_1} = A(ud\mu)_i$. We easily see that

$$r_j^{-l_1}(b_{0,j}-\tilde{b}_{0,j}) = \Big(M(ud\mu)^{-1}A(ud\mu-\tilde{u}d\mu)\Big) + \Big(M(ud\mu)^{-1}M(\tilde{u}d\mu-ud\mu)(r_j^{-l_1}b_{0,j})\Big)$$

however by (3.14) we have both $\|M(ud\mu - \tilde{u}d\mu)\| \le C(N,r)\|u - \tilde{u}\|_1 \sum_i r_i^{l_1+l_2}$ and that $\|I - M^{-1}(ud\mu)\| \le C(N,r)\|u\|_1 \sum_i r_i^{l_1+l_2}$, while (3.15) gives us that $\|A(ud\mu - \tilde{u}d\mu)\| \le C(N,r)\sum_i r_i^{l_2}\|u - \tilde{u}\|_1$. In both cases we have used that the total variation of $u_{i,j} - \tilde{u}_{i,j}$ is bounded by $\|u_{i,j} - \tilde{u}_{i,j}\|_1$ and that writing $u_X = \int_X u$ we can calculate

$$\int |u_{i,j}-\tilde{u}_{i,j}| = \int_X \left|u_X^{-1}u(x) - \tilde{u}_X\tilde{u}(x)\right| \le \tilde{u}_X^{-1}(1+|u|_X u_X^{-1})\int |u-\tilde{u}| \le 8\|u-\tilde{u}\|_1$$

The conclusion is then that $r_j^{-l_1}|b_{0,j} - \tilde{b}_{0,j}| \le C(N,r)\|u - \tilde{u}\|_1 \sum_i r_i^{l_2}$. Substituting this into (3.21) and using (3.7) and (3.8) we find that on $X \setminus Y$

$$\begin{aligned}\left|G(v-\tilde{v})(x)\right| &\le C(N,r)\Big(\sum_{i=1}^{N} r_i^{l_2}\Big)\sup_{i,j}\left\|u_{i,j}d\mu - \tilde{u}_{i,j}d\mu\right\| + C(N,r)\Big(\sum_i r_i^{l_2}\Big)\|u-\tilde{u}\|_1 \\ &\le C(N,r)\Big(\sum_i r_i^{l_2}\Big)\|u-\tilde{u}\|_1\end{aligned}$$

because the total variation $\|u_{i,j}d\mu - \tilde{u}_{i,j}d\mu\|$ was already computed to be at most $8\|u - \tilde{u}\|_1$. On the rest of X we must use (3.6) instead of (3.7). The weaker estimate is easily computed to be

$$\left|G(v-\tilde{v})(x)\right| \le C(N,r)\|u-\tilde{u}\|_1 \tag{3.22}$$

From our estimates on $G(v - \tilde{v}) = \Psi u - \Psi \tilde{u}$ we see that Ψ is a contraction on L^1 if l_1 and l_2 are sufficiently large, because

$$\begin{aligned}\left\|\Psi u - \Psi \tilde{u}\right\|_1 &\leq \mu(X \setminus Y) C(N, r) \Big(\sum_i r_i^{l_2}\Big) \|u - \tilde{u}\|_1 + \mu(Y) C(N, r) \|u - \tilde{u}\|_1 \\ &\leq C(N, r) \Big(\mu(Y) + \sum_j r_j^{l_2}\Big) \left\|u - \tilde{u}\right\|_1\end{aligned}$$

It follows readily that Ψ has a unique fixed point in C and $\Psi^k u$ converges to this fixed point in L^1. From (3.22) this convergence is uniform, and we notice that the correct choice of l_2 provides $|u_0 - 1| = |\Psi u_0 - 1| \leq \epsilon$ on $X \setminus Y$ by (3.18).

It remains only to see that u_0 is a smooth bump function on X. Inductively suppose $\Psi^k u \in \operatorname{dom}(\Delta^k)$ and both $\Delta^j \Psi^k u \equiv 0$ on ∂X for $0 \leq j \leq k$ and $\partial_n \Delta^j \Psi^k u \equiv 0$ on ∂X for $0 \leq j \leq k - 1$. This is certainly true for $k = 0$. By construction, $\Delta \Psi^{k+1} u$ is a linear combination of rescaled copies of $\Psi^k u$ that have been extended by zero as in (3.9). Each of these functions is in $\operatorname{dom}(\Delta^k)$ by the matching conditions for the Laplacian, so we conclude that $\Psi^{k+1} u \in \operatorname{dom}(\Delta^{k+1})$. It is immediate that $\Delta^j \Psi^{k+1} u \equiv 0$ on ∂X for $1 \leq j \leq k + 1$ and $\partial_n \Delta^j \Psi^{k+1} u \equiv 0$ on ∂X for $1 \leq j \leq k$. By Lemma 3.7 we know also that $\Psi^{k+1} u$ and $\partial_n \Psi^{k+1} u$ vanish on ∂X, which closes the induction and establishes that $u_0 \in \operatorname{dom}(\Delta^\infty)$ and vanishes to infinite order on ∂X. □

4 A Borel theorem on p.c.f. fractals

The classical Borel theorem tells us that given any neighborhood of $x_0 \in \mathbb{R}$ and any prescribed sequence of values for u and its derivatives at x_0, we may construct a smooth function u with support in the neighborhood and the given sequence of derivatives at x_0. Using the smooth bump functions we have constructed, we now show that the same result holds at junction points of certain p.c.f. fractals. In what follows X is p.c.f. and self-similar under $\{F_j\}_{j=1}^N$ and the measure μ is self-similar with factors $0 < \mu_j < 1$, $\sum_1^N \mu_j = 1$, so that $\mu(F_w(X)) = \prod_{j=1}^m \mu_{w_j}$ when w is the word $w_1 \dots w_m$. The Dirichlet form is that associated to a regular self-similar harmonic structure with resistance renormalization factors $0 < r_j < 1$ for $j = 1, \dots, N$. Our arguments depend on the existence of smooth bumps as previously constructed. The crucial step is the existence of smooth functions with finitely many prescribed normal derivative values, which is established in the following lemma.

Lemma 4.1. *Given a boundary point $q \in V_0$ there are smooth functions f_l such that*

$$\begin{gathered}\Delta^k f_l(p) = 0 \quad \text{for all } k \geq 0 \text{ and all } p \in V_0 \\ \partial_n \Delta^k f_l(p) = 0 \quad \text{for all } k \geq 0 \text{ and all } p \in V_0 \setminus \{q\} \\ \partial_n \Delta^k f_l(q) = \delta_{lk}\end{gathered}$$

Proof. We begin with the case $l = 0$. If U is the smooth positive bump function on X produced in Theorem 2.1 we localize it near the boundary points of X at a scale m to be determined later. Define

$$U_j = \begin{cases} U \circ F_j^{-m} & \text{on } F_j^m(X) \\ 0 & \text{otherwise} \end{cases}$$

and observe from the matching conditions for the Laplacian that each U_j is smooth. Now apply the Dirichlet Green's operator G to these functions and form the linear combination

$$f = \sum_{j=1}^{N} a_j G(U_j)$$

with coefficients to be chosen. It is clear from the properties of U that $\Delta^k f = 0$ on V_0 for all $k \geq 0$ and that $\partial_n \Delta^k f = 0$ on V_0 if $k \geq 1$. Moreover the Gauss-Green formula yields values of the normal derivatives at the points $q_i \in V_0$

$$\partial_n f(q_i) = -\sum_{j=1}^{N} a_j \int_X h_i U_j$$

where h_i is the harmonic function on X with boundary values $h_i(q_j) = \delta_{ij}$. In order that there be coefficients a_j such that f has the properties asserted for f_0 it then suffices that we can invert the matrix with entries $A_{ij} = \int h_i U_j$. We use the fact that

$$|h_i| \leq r_j^m \quad \text{on} \quad F_j^m(X) \quad \text{for} \quad j \neq i$$
$$|h_i - 1| \leq r_i^m \quad \text{on} \quad F_i^m(X)$$

which follow from the estimates on the oscillation of a harmonic function that were mentioned at the end of the introduction. Using this we calculate

$$|A_{ij}| = \left| \int h_i U_j \right| \leq r_j^m \mu_j^m \int_X U \quad \text{for } j \neq i$$
$$\left| A_{ii} - \mu_i^m \int_X U \right| = \left| \int (h_i - 1) U_i \right| \leq r_i^m \mu_i^m \int_X U \tag{4.1}$$

Let D be the diagonal matrix with entries $D_{ii} = \mu_i^m \int_X U$. Then we readily compute $(AD^{-1})_{ij} = \mu_j^{-m} (\int U)^{-1} A_{ij}$ is close to the identity if m is large. Indeed, by (4.1) we have $|(I - AD^{-1})_{ij}| \leq \rho^m$ with $\rho = \max_i r_i$, so that AD^{-1} is invertible provided m is sufficiently large.

We proceed by induction on l, with an almost unchanged argument. Suppose the functions f_l for $l \leq L - 1$ have been constructed as linear combinations of the form

$$f_l = \sum_{n=1}^{l} \sum_{j=1}^{N} a_{jn} G^{(n+1)}(U_j) \quad l \leq L - 1 \tag{4.2}$$

and consider the function

$$f = \sum_{j=1}^{N} a_j G^{(L+1)}(U_j)$$

so that $\Delta^k f = 0$ on V_0 for all k and $\partial_n \Delta^k f = 0$ on V_0 for $k \geq L + 1$. When $k = L$ we have

$$\partial_n \Delta^L f(q_i) = -\sum_{j=1}^{N} a_j A_{ij}$$

where A_{ij} is as before, so we may select a_j to obtain $\partial_n \Delta^L f(q) = 1$ and $\partial_n \Delta^L f(p) = 0$ at other points $p \in V_0$. Subtracting an appropriate linear combination of the f_l for $l \leq L - 1$ we obtain the desired f_L in the form of (4.2). □

With this in hand it is simple to deal with finitely many values of the Laplacian at a boundary point $q \in V_0$.

Lemma 4.2. *Given a boundary point $q \in V_0$ there are smooth functions g_l such that*

$$\begin{aligned} \partial_n \Delta^k g_l(p) = 0 \quad &\textit{for all } k \geq 0 \textit{ and all } p \in V_0 \\ \Delta^k g_l(p) = 0 \quad &\textit{for all } k \geq 0 \textit{ and all } p \in V_0 \setminus \{q\} \\ \Delta^k g_l(q) = \delta_{lk} & \end{aligned}$$

Proof. Let h be the harmonic function which is 1 at q and 0 at all other points of V_0. Clearly $\Delta^k h \equiv 0$ for all $k \geq 1$ and therefore also $\partial_n \Delta^k h(p) = 0$ for all $k \geq 1$ and $p \in V_0$. For each $p \in V_0$ let $f_{0,p}$ be the function constructed in Lemma 4.1 with non-vanishing normal derivative at p. It is clear that $g_0 = h - \sum_{p \in V_0} (\partial_n h(p)) f_{0,p}$ has the desired properties, so we have found the first of our functions. To obtain the others we simply apply the Dirichlet Green's operator G. Notice that $\Delta^k G^l h(p) = \delta_{k,l}\delta_{p,q}$ for all k and all $p \in V_0$, and also $\partial_n \Delta^k G^l h(p) = 0$ for all $k \geq l + 1$. To obtain g_k it remains only to subtract off all normal derivatives that occur for $0 \leq k \leq l$ using the functions from Lemma 4.1. □

The proof of a Borel-type theorem from the above lemmas is standard. All that is needed is information about how scaling the support of a function changes its Laplacian and normal derivatives. Recall that for a Dirichlet form associated to a regular self-similar resistance, both the Laplacian and the normal derivative may be obtained as renormalized limits of corresponding quantities defined on the approximating graphs (Section 3.7 of [10]). In particular, pre-composition with the map F_i^{-1} rescales the k-th power of the Laplacian by $(\mu_i r_i)^{-k}$ and its normal derivative by $\mu_i^{-k} r_i^{-k-1}$. For this reason, if $q = q_i$ is the boundary point of interest we define

$$f_{l,m} = \begin{cases} \mu_i^{mk} r_i^{m(k+1)} f_l \circ F_i^{(-m)} & \text{on } F_i^m \\ 0 & \text{otherwise} \end{cases}$$

$$g_{l,m} = \begin{cases} (\mu_i r_i)^{mk} g_l \circ F_i^{(-m)} & \text{on } F_i^m \\ 0 & \text{otherwise} \end{cases}$$

so that we have for all k

$$\begin{aligned} \Delta^k f_{l,m}(q) &= 0 & \Delta^k g_{l,m}(q) &= \delta_{lk} \\ \partial_n f_{l,m}(q) &= \delta_{lk} & \partial_n \Delta^k g_{l,m}(q) &= 0 \end{aligned} \tag{4.3}$$

but the L^∞ norms of the lower order derivatives have decreased and those of the higher order derivatives have increased.

$$\|\Delta^k f_{l,m}\|_\infty = \mu_i^{m(k-l)} r_i^{m(k+1-l)} \|\Delta^k f_l\|_\infty \tag{4.4}$$

$$\|\Delta^k g_{l,m}\|_\infty = (\mu_i r_i)^{m(k-l)} \|\Delta^k g_l\|_\infty \tag{4.5}$$

With this in hand we can easily prove our version of the Borel theorem.

Theorem 4.3. *Let $q \in V_0$ be fixed, and Ω be an open neighborhood of q. Given a jet $\rho = (\rho_0, \rho_1, \dots)$ of values for powers of the Laplacian and $\sigma = (\sigma_0, \sigma_1, \dots)$ of values for their normal derivatives, there is a smooth function f with support in Ω and both $\Delta^k f(q) = \rho_k$ and $\partial_n \Delta^k f(q) = \sigma_k$ for all k.*

Proof. We give the usual proof that it is possible to define f by the series

$$f = \sum_l \Big(\rho_l g_{l,m_l} + \sigma_l f_{l,n_l} \Big) \tag{4.6}$$

for an appropriate choice of m_l and n_l.

Let $m_0 = n_0$ be sufficiently large that $F_i^{m_0}(X) \subset \Omega$. For each l choose $m_l \geq m_0$ so large that

$$\left\| \rho_l \Delta^k g_{l,m_l} \right\|_\infty \leq 2^{k-l-1} \quad \text{for } 0 \leq k \leq l-1$$

using the scaling estimate (4.5). Similarly use the scaling relation (4.4) to take $n_l \geq m_0$ such that

$$\left\| \sigma_l \Delta^k f_{l,n_l} \right\|_\infty \leq 2^{k-l-1} \quad \text{for } 0 \leq k \leq l-1$$

Then for fixed k we have

$$\left\| \sum_{l=k+1}^{\infty} \Delta^k \Big(\rho_l g_{l,m_l} + \sigma_l f_{l,n_l} \Big) \right\|_\infty \leq \sum_{k+1}^{\infty} 2^{k-l} \leq 1$$

so that all powers of the Laplacian applied to (4.6) produce L^∞ convergent series. It follows that f as defined in (4.6) is smooth and has support in Ω. By (4.3) it has the desired jet, so the result follows. □

We remark that for any $\epsilon > 0$ we could replace the bounds 2^{k-l-1} in the proof with $\epsilon 2^{k-l-1}$. It follows that we can define f by (4.6) and have the estimate

$$\left\| \Delta^k f \right\|_\infty \leq C(k, \Omega) \sum_{l=0}^{k} (|a_l| + |b_l|) + \epsilon \tag{4.7}$$

where $C(k, \Omega)$ does not depend on the jet we prescribe.

It is also clear that the result of the theorem may be transferred to any junction point $F_w(q)$ and cell $F_w(X)$ in X, simply by modifying the desired jet to account for the effect of composition with F_w, solving for f on X, and defining the new function to be $f \circ F_w^{-1}$ on the cell. We record a version of this that will be useful later; note that in the following we use the notation ∂_n for the normal derivative with respect to the cell $F_w(X)$.

Corollary 4.4. *Let $F_w(q)$ be a junction point in X. Given a jet $(\rho_0, \rho_1, \dots), (\sigma_1, \sigma_2, \dots)$ there is a smooth function f on $F_w(X)$ that has $\Delta^k f(p) = \partial_n \Delta^k f(p) = 0$ at all points $p \in \partial F_w(X)$ such that $p \neq F_w(q)$, and satisfies $\Delta^k f(F_w(q)) = \rho_k$ and $\partial_n \Delta^k f(F_w(q)) = \sigma_k$ for all k.*

5 Additive Partitions of Functions

The results of [4] show that multiplication is not generally a good operation on functions in $\mathrm{dom}(\Delta)$. In particular, for X a p.c.f. fractal with self-similar measure and regular self-similar harmonic structure it is generically the case that if $u \in \mathrm{dom}(\Delta)$ then $u^2 \notin \mathrm{dom}(\Delta)$. In such a situation there is no hope of using a smooth partition of unity to localize problems in the classical manner. Instead we provide a simple method for making a smooth decomposition of $f \in \mathrm{dom}(\Delta^\infty)$ using Theorem 4.3.

Theorem 5.1. *Let $\cup_\alpha \Omega_\alpha$ be an open cover of X and $f \in \mathrm{dom}(\Delta^\infty)$. There is a decomposition $f = \sum_{k=1}^K f_k$ in which each k has a corresponding α_k such that f_k is smooth on X and supported in Ω_{α_k}.*

Proof. Compactness of X allows us to reduce to the case of a finite cover $\cup_1^K \Omega_{\alpha_k}$ for which there is no sub-collection that covers X. We write Ω_k for Ω_{α_k}, and construct the functions f_k inductively. At the k-th stage we suppose there are functions $f_1, \dots, f_{k-1}$ with the properties asserted in the lemma, and that function $g_{k-1} = f - \sum_{l=1}^{k-1} f_l$ is smooth on X and vanishes identically on a neighborhood Π_{k-1} of $\Omega \setminus \left(\cup_{j=k}^K \Omega_j\right)$. In the base case $k = 1$ this assumption is trivial, and it is clear that the theorem follows immediately from the induction. We have therefore reduced to the case where our cover consists of the two sets Ω_k and $\tilde{\Omega}_k = \cup_{j=k+1}^K \Omega_j$, because the induction is complete once we have f_k as in the lemma such that g_k is identically zero on a neighborhood Π_k of $\Omega \setminus \tilde{\Omega}_k$.

For a scale m and $x \in X$, define the m-scale open neighborhood of x to be the interior of the unique m-cell containing x if $x \notin V_m$, and to be the union $\{x\} \cup (\cup_w F_w(X) \setminus \partial F_w(X))$ if $x = F_w(q_i)$ is a junction point. By Section 1.3 of [10], the m-scale open neighborhoods form a fundamental system of open neighborhoods of x. At each x in Ω_k there is a largest m such that the m-scale neighborhood of x is contained in Ω_k. The collection of all such largest neighborhoods of points of Ω_k is an open cover of the compact set $\mathrm{Sppt}(g_{k-1}) \setminus \tilde{\Omega}_k$. We use Λ_k to denote the union over a finite subcover.

Clearly Λ_k has finitely many boundary points. Let those boundary points that are also in $\tilde{\Omega}_k$ be $x_1, \dots, x_J$, and take at each a finite collection of cells $\{C_{i,j}\}_{i=1}^{I_j}$ having x_j in their boundary. We require that $\Lambda_k \cup \left(\cup_{i=1}^{I_j} C_{i,j}\right)$ contains a neighborhood of x_j, that all of the $C_{i,j}$ lie entirely within Ω_k and none intersect Λ_k, and that $C_{i,j} \cap C_{i',j'}$ is empty unless $j = j'$, in which case it contains only x_j. On each cell we apply Corollary 4.4 to find functions $h_{i,j}$ that match $g_{k-1}(x_j)$ and all powers of its Laplacian at x_j, and such that the sum $\sum_i h_{i,j}$ has normal derivatives that cancel $\partial_n \Delta^n g_{k-1}(x_j)$ at x_j for all n. Thus $\sum_{i=1}^{I_j} h_{i,j}$ matches g_{k-1} in the sense of the matching condition and vanishes to infinite order at the other boundary points of $\cup_{i=1}^{I_j} C_{i,j}$. The matching condition implies that $f_k = g_{k-1}|_{\Lambda_k} + \sum_j \sum_i h_{i,j}$ is smooth. It is clearly supported on Ω_k and equal to f on the closure $\overline{\Lambda}_k$ of Λ_k, so $g_k = g_{k-1} - f_k$ is zero on a neighborhood Π_k of $\Omega \setminus \tilde{\Omega}_k$, which completes the induction and the proof. □

References

[1] Martin T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, volume 1690 of *Lecture Notes in Math.*, pages 1–121. Springer, Berlin, 1998.

[2] Martin T. Barlow and Richard F. Bass. The construction of Brownian motion on the Sierpiński carpet. *Ann. Inst. H. Poincaré Probab. Statist.*, 25(3):225–257, 1989.

[3] Martin T. Barlow and Richard F. Bass. Brownian motion and harmonic analysis on Sierpinski carpets. *Canad. J. Math.*, 51(4):673–744, 1999.

[4] Oren Ben-Bassat, Robert S. Strichartz, and Alexander Teplyaev. What is not in the domain of the Laplacian on Sierpinski gasket type fractals. *J. Funct. Anal.*, 166(2):197–217, 1999.

[5] Nitsan Ben-Gal, Abby Shaw-Krauss, Robert S. Strichartz, and Clint Young. Calculus on the sierpinski gasket II: Point singularities, eigenfunctions, and normal derivatives of the heat kernel. *Trans. Amer. Math. Soc.*, posted on April 17, 2006, PII: S 0002-9947(06)04056-6 (to appear in print).

[6] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[7] Pat J. Fitzsimmons, Ben M. Hambly, and Takashi Kumagai. Transition density estimates for Brownian motion on affine nested fractals. *Comm. Math. Phys.*, 165(3):595–620, 1994.

[8] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[9] P. Edward Herman, Roberto Peirone, and Robert S. Strichartz. *p*-energy and *p*-harmonic functions on Sierpinski gasket type fractals. *Potential Anal.*, 20(2):125–148, 2004.

[10] Jun Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[11] Jun Kigami. Harmonic analysis for resistance forms. *J. Funct. Anal.*, 204(2):399–444, 2003.

[12] Takashi Kumagai. Short time asymptotic behaviour and large deviation for Brownian motion on some affine nested fractals. *Publ. Res. Inst. Math. Sci.*, 33(2):223–240, 1997.

[13] Tom Lindstrøm. Brownian motion on nested fractals. *Mem. Amer. Math. Soc.*, 83(420):iv+128, 1990.

[14] Robert Meyers, Robert S. Strichartz, and Alexander Teplyaev. Dirichlet forms on the Sierpiński gasket. *Pacific J. Math.*, 217(1):149–174, 2004.

[15] Jonathan Needleman, Robert S. Strichartz, Alexander Teplyaev, and Po-Lam Yung. Calculus on the Sierpinski gasket. I. Polynomials, exponentials and power series. *J. Funct. Anal.*, 215(2):290–340, 2004.

[16] Kasso A. Okoudjou, Luke G. Rogers, and Robert S. Strichartz. Generalized eigenfunctions and a Borel theorem on the Sierpinski gasket. *Submitted to Canad. Math. Bull.*

[17] Roberto Peirone. Convergence and uniqueness problems for Dirichlet forms on fractals. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 3(2):431–460, 2000.

[18] C. Sabot. Existence and uniqueness of diffusions on finitely ramified self-similar fractals. *Ann. Sci. École Norm. Sup. (4)*, 30(5):605–673, 1997.

[19] Robert S. Strichartz. *Differential Equations on Fractals: A Tutorial*. Princeton University Press, to appear 2006.

[20] Robert S. Strichartz. Analysis on products of fractals. *Trans. Amer. Math. Soc.*, 357(2):571–615 (electronic), 2005.