Math 6310, Fall 2019
Homework 1

1. Let H and K be subgroups of a group G. Recall that

$$HK := \{hk \mid h \in H,\ k \in K\}.$$

   (a) Give an example in which $H \cup K$ is not a subgroup of G.
   (b) Give an example in which HK is not a subgroup of G.
   (c) Show that the following statements are equivalent.
      i. $HK = \langle H \cup K \rangle$.
      ii. HK is a subgroup of G.
      iii. $HK = KH$.
   (d) Show that if $H \subseteq N_G(K)$ then $HK = KH$.
   (e) Show that the converse to (1d) does not hold.
   (f) Suppose H and K are both normal in G and $H \cap K = \{1\}$. Show that $hk = kh$ for all $h \in H$, $k \in K$.

2. Let $(I, \leq)$ be a partially order set such that for any $i, j \in I$ there exists $k \in I$ such that

$$i \leq k \quad \text{and} \quad j \leq k.$$

   Let $\{G_i\}_{i \in I}$ be a family of subgroups of a group G such that if $i \leq j$ then $G_i \subseteq G_j$.

   (a) Show that in this case $\bigcup_{i \in I} G_i \leq G$.
   (b) Let μ_n be the group of n-th roots of unity in $\mathbb{C}$. Deduce that

$$\mu_\infty \ := \ \bigcup_{n \geq 1} \mu_n$$

   is a subgroup of S^1.

3. (a) Let A, B, C be subgroups of a group D such that $B, C \trianglelefteq D$ and $A = B \cap C$. Use the isomorphism theorems to show that

$$\frac{D/B}{C/A} \cong \frac{D/C}{B/A}.$$

   First explain why each quotient makes sense.

   (b) Let $\mathbb{F}$ be a field and n a positive integer. If necessary, look up the definition of the projective general and projective special linear groups

$$\mathrm{PGL}(n, \mathbb{F}) \quad \text{and} \quad \mathrm{PSL}(n, \mathbb{F}).$$

   Let

$$(\mathbb{F}^\times)^{(n)} := \{x \in \mathbb{F}^\times \mid \text{ there is } y \in \mathbb{F}^\times \text{ such that } x = y^n\}$$

   be the groups of n-th powers in $\mathbb{F}^\times$. Deduce that

$$\mathrm{PGL}(n, \mathbb{F})/\mathrm{PSL}(n, \mathbb{F}) \cong \mathbb{F}^\times/(\mathbb{F}^\times)^{(n)}.$$

4. (a) The second isomorphism law states that if H normalizes N, then

$$H/H\cap N \to HN/N, \quad h(H\cap N)\mapsto hN$$

is an isomorphism. Describe the inverse isomorphism explicitly.

(b) Describe the isomorphism in the Butterfly Lemma explicitly.

5. Let G be a finite group.

(a) Let $f \in \mathrm{End}(G)$. Prove that, for n sufficiently large, the subgroups $N := \mathrm{Ker}(f^n)$ and $H := \mathrm{Im}(f^n)$ are independent of n and satisfy

$$G = NH \quad \text{and} \quad N \cap H = \{1\}.$$

(This says that G is the semidirect product of N and H.) Moreover, f induces an automorphism of H.

(b) Deduce that if G cannot be decomposed as a semidirect product in a nontrivial way, then every endomorphism either is nilpotent (i.e., some power of it is trivial) or is an automorphism.

6. Consider the dihedral group of order $2n$.

(a) When n is even, find two subnormal series of length 2 for which $\mathbb{Z}_2$ is one of the slices, but it appears first in one series and last in the other.

(b) Are there such series when n is odd?

7. Let G be a group with a composition series and $H \trianglelefteq G$.

(a) Show that G has a composition series in which H is one of the terms. Deduce that H and G/H have composition series.

(b) The length of some (every) composition series of G is denoted $\ell(G)$. Show that $\ell(G) = \ell(H) + \ell(G/H)$.

(c) If K is another normal subgroup of G, show that

$$\ell(HK) = \ell(H) + \ell(K) - \ell(H \cap K).$$

8. (a) Let $\rho = (a_1, \ldots, a_r)$ be an r-cycle and σ a permutation in S_n. Show that

$$\sigma\rho\sigma^{-1} = \big(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)\big).$$

(b) Describe the conjugacy classes in S_n.

9. (a) Compute the conjugacy classes in A_5.

(b) Prove that A_5 is simple.

10. Let Ω be a set. An Ω-*group* is a group G together with a map

$$\Omega \times G \to G, \quad (\omega, g) \mapsto {}^{\omega}g$$

such that

$${}^{\omega}(gh) = {}^{\omega}g\,{}^{\omega}h$$

for all $\omega \in \Omega$, $g, h \in G$. Note that this is equivalent to a map $\Omega \to \mathrm{End}(G)$, where $\mathrm{End}(G)$ denotes the set of all homomorphisms $G \to G$. Thus, G is a group with a collection of endomorphisms indexed by Ω. An Ω-group is also called a *group with operators*.

(a) Define suitable notions of Ω-subgroup and homomorphism of Ω-groups.

(b) An Ω-subgroup of an Ω-group G is *normal* if it is normal as a subgroup of G. Let N be such a subgroup. Show that G/N is an Ω-group in such a way that the canonical projection $G \twoheadrightarrow G/N$ is a homomorphism of Ω-groups.

(c) Briefly review the isomorphism laws and note that they hold in the context of Ω-groups.

(d) An Ω-group G is simple if it is nontrivial and the only normal Ω-subgroups are $\{1\}$ and G. An Ω-composition series of G is a subnormal series whose slices are simple Ω-groups. Review the Butterfly Lemma, Schreir's Refinement Theorem, and the Jordan-Hölder Theorem, and note that they hold in the context of Ω-groups.

11. Let G be a group.

    (a) Show that in each of the following cases, G is an Ω-group.

        i. $\Omega = \emptyset$ (with no operators).
        ii. $\Omega = G$, with ${}^g h := ghg^{-1}$.
        iii. $\Omega = \mathrm{Aut}(G)$, with ${}^\sigma h := \sigma(h)$.

    (b) In each of the previous cases, describe explicitly the notions of Ω-subgroup and normal Ω-subgroup.

    (c) In each of the previous cases, describe the notion of isomorphism of Ω-groups in terms of the groups $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$.

    (d) In case ii, an Ω-composition series is called a *chief* series. Describe this notion explicitly.

12. Let V be a vector space over a field $\mathbb{F}$. Let G be the additive group of V and $\Omega = \mathbb{F}$.

    (a) Note that G is an Ω-group with ${}^\lambda v := \lambda \cdot v$ for $\lambda \in \mathbb{F}$ and $v \in V$.

    (b) What is an Ω-subgroup of G? What is a homomorphism of Ω-groups $G \to H$, if both G and H arise from vector spaces V and W over $\mathbb{F}$ as above?

    (c) When is G a simple Ω-group? What is an Ω-composition series of G?

    (d) Deduce from Exercise 7c (for Ω-groups) a familiar result from linear algebra about dimensions of subspaces.

    Note: a module M over a ring R similarly gives rise to an R-group structure on the additive group of M. The Jordan-Hölder Theorem for Ω-groups yields in this manner the Jordan-Hölder Theorem for modules.

13. Let G be an Ω-group. Prove the following statements, or give a counterexample.

    (a) The commutator subgroup $[G, G]$ is an Ω-subgroup.

    (b) The center $Z(G)$ is an Ω-subgroup.

14. Let x, y and z be integers such that x divides z. Prove that

$$\mathrm{lcm}\big(x, \gcd(y, z)\big) = \gcd\big(\mathrm{lcm}(x, y), z\big).$$

15. (a) Give an example of two nonisomorphic groups with isomorphic composition factors.

(b) Let $S_1, \ldots, S_n$ be a collection of simple groups (not necessarily nonisomorphic). Construct a group with those groups for composition factors.

16. Let G be a group whose only subgroups are $\{1\}$ and G. Show that G is either trivial or cyclic of prime order.

17. Let G be a finite abelian group and p a prime divisor of $|G|$. Prove that G contains an element of order p, without appealing to the structure theorem for finite abelian groups.

    Hint: choose a proper nontrivial subgroup (when possible) and argue by induction.

    This exercise proves a result we will use in class when discussing p-groups. *Cauchy's Theorem* says that the result holds for all finite groups, not just the abelian ones. In class we will deduce Cauchy's Theorem from the Sylow Theorems.