ON THE ASSOCIATIVE ANALOG OF LIE BIALGEBRAS

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Abstract. An infinitesimal bialgebra is at the same time an associative algebra and coalgebra in such a way that the comultiplication is a derivation [A1]. This paper continues the basic study of these objects, with emphasis on the connections with the theory of Lie bialgebras.

It is shown that non degenerate antisymmetric solutions of the associative Yang-Baxter equation are in one to one correspondence with non degenerate cyclic 2-cocycles. The associative and classical Yang-Baxter equation are compared: it is studied when a solution to the first is also a solution to the second. Necessary and sufficient conditions for obtaining a Lie bialgebra from an infinitesimal one are found, in terms of a canonical map that behaves simultaneously as a commutator and a cocommutator. The class of balanced infinitesimal bialgebras is introduced; they have an associated Lie bialgebra. Several well known Lie bialgebras are shown to arise in this way. The construction of Drinfeld's double from [A1], for arbitrary infinitesimal bialgebras, is complemented with the construction of the *balanced double*, for balanced ones. This construction commutes with the passage from balanced infinitesimal bialgebras to Lie bialgebras.

1. Introduction

An infinitesimal bialgebra (abbreviated ϵ-bialgebra) is a triple (A, m, Δ) where (A, m) is an associative algebra (possibly without unit), (A, Δ) is a coassociative coalgebra (possibly without counit) and, for each $a, b \in A$,

$$\Delta(ab) = ab_1 \otimes b_2 + a_1 \otimes a_2 b \ . \tag{1.1}$$

In other words, Δ is required to be a derivation of the associative algebra A with values on the A-bimodule $A \otimes A$. The notion of ϵ-bialgebra is manifestly analogous to that of a Lie bialgebra.

Gian-Carlo Rota argued in favor of the study of ϵ-bialgebras in several occasions, starting perhaps with [J-R, section XII], which precedes the introduction of Lie bialgebras by Drinfeld in the early 80's. Apparently, such program remained uncontested until recently, when new motivations were found, coming on one hand from interesting applications to combinatorics [A2, E-R] and on the other from the close relation to the theory of Lie bialgebras.

In [A1], the basic theory of ϵ-bialgebras was presented. This includes the introduction of several examples and the notions of antipode, Drinfeld's double and associative Yang-Baxter equation. The study of ϵ-bialgebras is continued in the present paper, with special emphasis on the analogy and connection with Lie bialgebras. The results obtained are summarized below, after the introduction of the necessary notation. The contents are described in more detail at the beginning of each section.

Notation. All vector spaces and algebras are over a fixed field k. $M_n(k)$ denotes the algebra of matrices. Sum symbols are often omitted from Sweedler's notation: we write $\Delta(a) = a_1 \otimes a_2$ when Δ is a coassociative comultiplication.

Date: May 2000.

2000 *Mathematics Subject Classification.* Primary: 16W30, 16W25.

Key words and phrases. Hopf algebras, Lie bialgebras, derivations, Yang-Baxter equation, Drinfeld double.

Research supported by a Postdoctoral Fellowship at the Centre de Recherches Mathématiques and Institut des Sciences Mathématiques, Montréal, Québec.

Let A be an associative algebra, possibly non unital. The space $A\otimes A$ is viewed as an A-bimodule via

$$a\cdot(u\otimes v)=au\otimes v \text{ and } (u\otimes v)\cdot a=u\otimes va\ .$$

Let $r\in A\otimes A$. The principal derivation defined by r is the map $\Delta_r : A\to A\otimes A$ defined by

$$\Delta_r(a)=a\cdot r-r\cdot a\ .$$

The symmetric and antisymmetric parts are denoted by

$$r^+=\frac{1}{2}\Big(r+\tau(r)\Big)\in S^2(A) \text{ and } r^-=\frac{1}{2}\Big(r-\tau(r)\Big)\in\Lambda^2(A)\ ,$$

where $\tau : A\otimes A\to A\otimes A$ is the switch $\tau(u\otimes v)=v\otimes u$. Consider the elements

$$\mathbf{A}(r)=r_{13}r_{12}-r_{12}r_{23}+r_{23}r_{13}\ ,$$
$$\mathbf{C}(r)=[r_{13},r_{12}]+[r_{23},r_{12}]+[r_{23},r_{13}]\ .$$

These belong to $A\otimes A\otimes A$ and are well defined even if A is non unital. The associative Yang-Baxter equation is [A1, section 5]

(AYB) $$\mathbf{A}(r)=0\ .$$

It is analogous to the classical Yang-Baxter equation

(CYB) $$\mathbf{C}(r)=0\ .$$

According to [A1, proposition 5.1], a principal derivation Δ_r is coassociative if and only if $\mathbf{A}(r)$ is an A-invariant element of the A-bimodule $A\otimes A\otimes A$. In this case one says that (A,r) is a coboundary ϵ-bialgebra. If the stronger condition (AYB) is satisfied, one says that (A,r) is a quasitriangular ϵ-bialgebra.

Contents. In section 2, antisymmetric solutions of the associative Yang-Baxter equation are studied. The main result, proposition 2.7, establishes a one to one correspondence between such solutions and those subalgebras of A carrying a non degenerate cyclic 2-cocycle. As an illustration, all antisymmetric solutions of (AYB) in $M_2(\mathbb{C})$ are found. An algebra carrying a non degenerate cyclic 2-cocycle is called *antisymmetric*, in this paper. Examples and basic properties of such algebras are given throughout the section.

In section 3, the associative and classical Yang-Baxter equation are compared, beyond the obvious formal analogy. It is shown that a solution r of (AYB) for which the symmetric part r^+ is invariant is also a solution of (CYB) (theorem 3.5). This allows us to obtain solutions of (CYB) from the solutions of (AYB) found in section 2. It is shown that some solutions of (CYB) by Belavin and Drinfeld arise in this way.

There is an obvious way of attempting to construct a Lie bialgebra from an ϵ-bialgebra, namely by endowing it with the commutator bracket and cocommutator cobracket. The necessary compatibilty condition between the two is, however, not always satisfied. In section 4, this situation is studied in detail. A canonical map $\mathbf{B} : A\otimes A\to A\otimes A$, defined on any ϵ-bialgebra A, is introduced. It is called the *balanceator*, for its basic properties are analogous to those of the commutator of an associative algebra, or the cocommutator of a coassociative coalgebra, simultaneously. These are given in proposition 4.3. An ϵ-bialgebra is called *balanced* if its balanceator is zero. This guarantees that passing to the commutator and cocommutator results in a Lie bialgebra. Two constructions of balanced ϵ-bialgebras from arbitrary ones are given (proposition 4.6). These are dual to each other and analogous to the construction of the center of an associative algebra. An important class of balanced ϵ-bialgebras is provided by quasitriangular ϵ-bialgebras (A,r) for which r^+ is A-invariant (proposition 4.7). This "explains" the result of corollary 3.7, about the passage from coboundary or quasitriangular ϵ-bialgebras to the respective classes of Lie bialgebras.

In section 5, the constructions of Drinfeld's doubles for ϵ-bialgebras and Lie bialgebras are compared. The Drinfeld double $D(A)$ of a finite dimensional ϵ-bialgebra A carries a canonical associative form, but unlike the case of Lie bialgebras, this fails to be non degenerate. Theorem 5.5 establishes a connection between the radical of this form and the fact that A be balanced. When A is balanced, the radical of the form admits a particularly simple description, and the quotient of $D(A)$ by this ideal turns out to be again a balanced ϵ-bialgebra, called the *balanced Drinfeld double* and denoted $D_b(A)$. This is perhaps one of the main constructions of the paper. As a vector space, $D_b(A) = A \oplus A^*$, and the multiplication in $D_b(A)$ is

$$(a, f) \cdot (b, g) = (ab + a \leftarrow g + f \rightarrow b,\ a \rightarrow g + f \leftarrow b + fg) \ ,$$

where the arrows denote the natural actions of A and A^* on each other. While $D(A)$ is a quasitriangular ϵ-bialgebra that is universally attached to any ϵ-bialgebra A, $D_b(A)$ is a balanced quasitriangular ϵ-bialgebra that is universally attached to any balanced ϵ-bialgebra A, as spelled out it theorem 5.9. The balanced double is thus strictly analogous to the double of a Lie bialgebra, but is defined only for balanced ϵ-bialgebras. This analogy is made more precise by proposition 5.12, where it is shown that the doubles constructions commute with the functor from balanced ϵ-bialgebras to Lie bialgebras.

Acknowledgements. I would like to thank Steve Chase and André Joyal for fruitful conversations.

2. Antisymmetric solutions of the associative Yang-Baxter equation

There is a one to one correspondence between non degenerate antisymmetric solutions of the classical Yang-Baxter equation in a Lie algebra $\mathfrak{g}$ and non degenerate 2-cocycles $\mathfrak{g}\otimes\mathfrak{g} \rightarrow k$ [C-P, corollary 2.2.4]. This can be used to classify all antisymmetric solutions in $\mathfrak{g}$ [C-P, proposition 2.2.6]. In this section we obtain the analogous results for the associative Yang-Baxter equation. Non degenerate antisymmetric solutions of the associative Yang-Baxter equation in A are in one to one correspondence with non degenerate cyclic 2-cocycles $A\otimes A \rightarrow k$ in the sense of Connes (also called 1-multitraces).

Let A be an associative, perhaps non unital, algebra. A cyclic 2-cocycle in the sense of Connes is an antisymmetric form $\omega : A\otimes A \rightarrow k$ such that

$$\omega(ab\otimes c) - \omega(a\otimes bc) + \omega(ca\otimes b) = 0 \quad \forall\ a, b, c \in A \ . \tag{2.1}$$

This corresponds to the original definition of cyclic cohomology by Connes [Con, pages 310-318]. It agrees with what is today called *periodic* cyclic cohomology, when $k \supseteq \mathbb{Q}$. See [Ros, 6.1.34 and 6.1.35]. An example of a 2-cocycle is the coboundary ω_f of a 1-form $f : A \rightarrow k$, defined by

$$\omega_f(a\otimes b) = f(ab - ba) \ .$$

We say that an antisymmetric tensor $r = \sum u_i\otimes v_i \in A\otimes A$ is non degenerate if the map

$$\tilde{r} : A^* \rightarrow A, \quad f \mapsto \sum f(u_i)v_i = -\sum u_i f(v_i) \tag{2.2}$$

is bijective. Similarly, an antisymmetric form $\omega : A\otimes A \rightarrow k$ is non degenerate if

$$\tilde{\omega} : A \rightarrow A^*, \quad \tilde{\omega}(a)(b) = \omega(a\otimes b) = -\omega(b\otimes a) \tag{2.3}$$

is bijective. Obviously, there is a one to one correspondence between such non degenerate tensors and forms via

$$\tilde{\omega} = \tilde{r}^{-1} \ . \tag{2.4}$$

Proposition 2.1. *Let A be a finite dimensional algebra. There is a one to one correspondence between non degenerate antisymmetric solutions $r \in A\otimes A$ of the associative Yang-Baxter equation and non degenerate Connes 2-cocycles $\omega : A\otimes A \rightarrow k$.*

Proof. Let $r \in A \otimes A$ be an antisymmetric non degenerate tensor. We will show that:

$$\mathbf{A}(r) = 0 \iff \tilde{r}^{-1} = \tilde{\omega} : A \to A^* \text{ is a derivation } \iff \omega \text{ is a Connes 2-cocycle.}$$

Here, A^* is viewed as an A-bimodule in the usual way:

$$(a \to f)(b) = f(ba) \text{ and } (f \leftarrow a)(b) = f(ab) \ .$$

Write $r = \sum u_i \otimes v_i$. For $f, g \in A^*$, we have

$$\begin{aligned}(f \otimes id \otimes g)\Big(\mathbf{A}(r)\Big) &= \sum f(u_i u_j) v_j g(v_i) - \sum f(u_i) v_i u_j g(v_j) + \sum f(u_j) u_i g(v_i v_j) \\ &\overset{(2.2)}{=} -\sum f\Big(\tilde{r}(g) u_j\Big) v_j + \tilde{r}(f)\tilde{r}(g) + \sum u_i g\Big(v_i \tilde{r}(f)\Big) \\ &= -\sum \Big(f \leftarrow \tilde{r}(g)\Big)(u_j) v_j + \tilde{r}(f)\tilde{r}(g) + \sum u_i \Big(\tilde{r}(f) \to g\Big)(v_i) \\ &\overset{(2.2)}{=} -\tilde{r}\Big(f \leftarrow \tilde{r}(g)\Big) + \tilde{r}(f)\tilde{r}(g) - \tilde{r}\Big((\tilde{r}(f) \to g\Big) \ .\end{aligned}$$

Since $\tilde{r} : A^* \to A$ is bijective we conclude that

$$\begin{aligned}\mathbf{A}(r) = 0 &\iff \forall\ f, g \in A^*,\ \tilde{r}(f)\tilde{r}(g) = \tilde{r}\Big((\tilde{r}(f) \to g\Big) + \tilde{r}\Big(f \leftarrow \tilde{r}(g)\Big) \\ &\iff \forall\ a, b \in A,\ \tilde{r}^{-1}(ab) = a \to \tilde{r}^{-1}(b) + \tilde{r}^{-1}(a) \leftarrow b \\ &\iff \tilde{r}^{-1} : A \to A^* \text{ is a derivation.}\end{aligned}$$

Evaluating the last equality on $c \in A$, and using (2.4), we obtain the additional equivalent condition:

$$\begin{aligned}&\forall\ a, b, c \in A,\ \tilde{\omega}(ab)(c) = \tilde{\omega}(b)(ca) + \tilde{\omega}(a)(bc) \\ &\overset{(2.3)}{\iff} \forall\ a, b, c \in A,\ \omega(ab \otimes c) = -\omega(ca \otimes b) + \omega(a \otimes bc)\end{aligned}$$

which says that ω is a Connes 2-cocycle as needed. ☐

Let $c_u : A \to A$ denote conjugation by an invertible element $u \in A$, $c_u(a) = uau^{-1}$. If $r \in A \otimes A$ is a (non degenerate, antisymmetric) solution of (AYB), then so is $(c_u \otimes c_u) r$. Two such solutions are said to be conjugate.

Proposition 2.2. *If two non degenerate antisymmetric solutions of the associative Yang-Baxter equation are conjugate, then the corresponding cocycles are cohomologous.*

Proof. Suppose r and s are two conjugate solutions, $s = (c_u \otimes c_u) r$. It follows readily that the corresponding cocycles are related by $\omega_r = \omega_s(c_u \otimes c_u)$. Hence, the cohomology classes of ω_r and ω_s are related by the conjugation action on periodic cyclic cohomology. Now, according to [Lod, proposition 4.1.3], this action is trivial. Hence ω_r and ω_s are cohomologous. ☐

Often, an algebra A is called *symmetric* if it possesses a non degenerate symmetric form $\langle\ ,\ \rangle : A \otimes A \to k$ that is associative, in the sense that

$$\langle ab, c \rangle = \langle a, bc \rangle \ \forall\ a, b, c \in A \ .$$

We will say that an algebra A is *antisymmetric* if it possesses a non degenerate Connes 2-cocycle.

Examples 2.3.

1. Let A be the two dimensional algebra with basis $\{x, y\}$ and multiplication

$$x^2 = 0,\ xy = 0,\ yx = x \text{ and } y^2 = y \ .$$

A is then an associative algebra; in fact, it can be realized as a subalgebra of $M_2(k)$ via $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Let $\omega : A \otimes A \to k$ be the form

$$\omega\Big((ax+by)\otimes(cx+dy)\Big) = ad - bc \ .$$

$\omega \sim \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is non degenerate, and it is a Connes 2-cocycle; in fact, it is the coboundary of $f : A \to k$, $f(x) = -1$, $f(y) = 0$. Thus A is an antisymmetric algebra. The corresponding non degenerate solution of (AYB) is

$$r = y \otimes x - x \otimes y \ .$$

(A, r) is the quasitriangular ϵ-bialgebra of [A1, example 5.4.5.e]

2. There are, up to isomorphism, only three antisymmetric algebras of dimension 2. These are the algebra of example 2.3.1, its opposite and the trivial algebra with zero multiplication. We sketch a proof next.

Consider a 2-dimensional algebra carrying a non degenerate antisymmetric solution r of (AYB). Choose a basis $\{x, y\}$ such that $r = y \otimes x - x \otimes y$ (or equivalently, $\omega \sim \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with respect to this basis). The associative Yang-Baxter equation for r (or the cocycle condition for ω) imposes some conditions on the structure constants of the algebra, from which it follows that the multiplication table must have the form:

$$x^2 = (a+b)x \qquad\qquad xy = dx + ay$$
$$y^2 = (c+d)y \qquad\qquad yx = cx + by$$

for some scalars a, b, c, d. Associativity leads to the following equations:

$$(*) \qquad ab = bc = cd = da = 0 \ .$$

Let us denote such an algebra by $A[a, b, c, d]$. The case $a = b = c = d = 0$ yields the trivial algebra with zero multiplication. Otherwise, we may assume that either $c \neq 0$ or $d \neq 0$, since, in general, $A[a, b, c, d] \cong A[c, d, a, b]$ via the map that interchanges x with y.

If $c \neq 0$ then $(*)$ implies $b = d = 0$, and $A[a, 0, c, 0] \cong A[0, 0, 1, 0]$ via $x' = cx - ay$ and $y' = \frac{y}{c}$. Note that $A[0, 0, 1, 0]$ is precisely the antisymmetric algebra of example 2.3.1.

If $d \neq 0$ then $(*)$ implies $a = c = 0$, and $A[0, b, 0, d] \cong A[0, 0, 0, 1]$ via $x' = cx - ay$ and $y' = \frac{y}{d}$. Note that $A[0, 0, 0, 1]$ is the opposite algebra of $A[0, 0, 1, 0]$. (They are not isomorphic.)

3. A higher dimensional antisymmetric algebra, which is a generalization of example 1, can be obtained as follows. Let $Z_m(k)$ denote the (non unital) subalgebra of $M_m(k)$ consisting of those matrices whose last row is equal to zero, and $f : Z_m(k) \to k$ the functional $f(\alpha) = \sum_{i=1}^{m-1} \alpha_{i,i+1}$, i.e. $f(\alpha)$ is the sum of the entries directly above the diagonal. The coboundary of f is then

$$\omega_f(\alpha \otimes \beta) = \sum_{i,j=1}^{m-1} \alpha_{i,j}\beta_{j,i+1} - \beta_{i,j}\alpha_{j,i+1} \ .$$

It is easy to see that ω_f is non degenerate and that the corresponding solution of (AYB) is

$$r = \sum_{i,j=1}^{m-1} \sum_{k=1}^{\max(i,j)} e_{i,i+j-k+1} \wedge e_{j,k}$$

where $e_{i,j}$ denote the elementary matrices and $x \wedge y = x \otimes y - y \otimes x$.

$Z_2(k)$ is the algebra of example 1 (f, ω and r have been multiplied by -1).

There is a simple way of constructing new Connes cocycles from old, which is reminiscent of the construction of a new Lie algebra as the tensor product of a Lie algebra and a commutative algebra.

Proposition 2.4. *Let A be an antisymmetric algebra with Connes cocycle ω_A and B a symmetric algebra with form $\langle\ ,\ \rangle_B$. Then $A\otimes B$ is an antisymmetric algebra with Connes cocycle*

$$\omega(a\otimes b, a'\otimes b') = \omega_A(a,a')\langle b,b'\rangle_B\ .$$

Proof. Non degeneracy is obvious. We check the cocycle condition (2.1). Since $\langle\ ,\ \rangle_B$ is associative and symmetric, we have

$$\langle b''b, b'\rangle_B = \langle b'', bb'\rangle_B = \langle bb', b''\rangle_B = \langle b, b'b''\rangle_B\ .$$

Therefore

$$\begin{aligned}\omega(a\otimes b\cdot a'\otimes b', a''\otimes b'') - \omega(a\otimes b, a'\otimes b'\cdot a''\otimes b'') + \omega(a''\otimes b''\cdot a\otimes b, a'\otimes b')\\ = \omega_A(aa',a'')\langle bb',b''\rangle_B - \omega_A(a,a'a'')\langle b,b'b''\rangle_B + \omega_A(a''a,a')\langle b''b,b'\rangle_B\\ = \Big(\omega_A(aa',a'') - \omega_A(a,a'a'') + \omega_A(a''a,a')\Big)\langle bb',b''\rangle_B = 0\ .\end{aligned}$$

□

Example 2.5. Let B be any symmetric algebra and $Z_m(B)$ the subalgebra of $M_m(B)$ consisting of those matrices with entries in B whose last row is zero. Then $Z_m(B)$ is an antisymmetric algebra, since it can be identified with the tensor product of B with the antisymmetric algebra $Z_m(k)$ of example 2.3.3.

In particular we may take $B = M_n(k)$, with the trace form. Then $Z_m(B)$ can be described as the subalgebra of $M_{mn}(k)$ consisting of those matrices whose last n rows are zero.

We recall an important result about solutions of (AYB).

Proposition 2.6. *Let A be a finite dimensional algebra and $r \in A\otimes A$ an arbitrary solution of the associative Yang-Baxter equation. When A^* is equipped with an appropriate dual structure, the map*

$$\tilde{r} : A^* \to A, \quad f \mapsto \sum f(u_i)v_i$$

is a morphism of ϵ-bialgebras.

Proof. See [A1, proposition 5.6]. □

From this we conclude that antisymmetric solutions of (AYB) correspond to antisymmetric subalgebras.

Proposition 2.7. *Let A be a finite dimensional algebra. There is a one to one correspondence between antisymmetric solutions in A of the associative Yang-Baxter equation and pairs (B,ω) where B is an (antisymmetric) subalgebra of A and ω a non degenerate Connes 2-cocycle on B.*

Proof. Let $r = \sum u_i\otimes v_i \in A\otimes A$ be an antisymmetric solution of (AYB). By proposition 2.6, $\mathsf{Im}(\tilde{r})$ is an ϵ-subbialgebra of A. By (2.2), $\mathsf{Im}(\tilde{r})$ is the subspace of A linearly spanned by either set $\{u_i\}$ or $\{v_i\}$. Hence, r is a non degenerate antisymmetric solution of (AYB) in $\mathsf{Im}(\tilde{r})$. By proposition 2.1, $\mathsf{Im}(\tilde{r})$ is an antisymmetric subalgebra of A.

Conversely, starting from an antisymmetric subalgebra B of A, proposition 2.1 yields an antisymmetric solution of (AYB) in B, and hence also in A. □

Example 2.8. As an application, we find all antisymmetric solutions of (AYB) in the algebra $M_2(\mathbb{C})$.

First of all, there are the antisymmetric subalgebras

$$Z_2(\mathbb{C}) = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \ / \ a, b \in \mathbb{C}\} \quad \text{and its transpose} \quad Z_2^t(\mathbb{C}) = \{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \ / \ a, b \in \mathbb{C}\}$$

of examples 2.3. These yield the solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and its transpose.

Suppose that r is another antisymmetric solution and let $B = \mathsf{Im}(\tilde{r})$ be the corresponding antisymmetric algebra. According to the remarks below, $\dim B$ must be even, and moreover B cannot be unital. Hence $\dim B$ is either 0 or 2. The first case corresponds to the trivial solution $r = 0$. In the second case, according to the result of example 2.3.2, B must be isomorphic to either $Z_2(\mathbb{C})$, its opposite algebra or the 2-dimensional algebra with zero multiplication. The opposite algebra of $Z_2(\mathbb{C})$ is obviously isomorphic to $Z_2^t(\mathbb{C})$. The 2-dimensional algebra with zero multiplication is not a subalgebra of $M_2(\mathbb{C})$ (being commutative, its elements could be simultaneously conjugated to nilpotent Jordan forms, but there is only one such form in $M_2(\mathbb{C})$), so this case is excluded.

Thus B is isomorphic to either $Z_2(\mathbb{C})$ or $Z_2^t(\mathbb{C})$. We claim that in each case the isomorphism is given by conjugation by an invertible element of $M_2(\mathbb{C})$. Consider for instance the first case. Let $\{x, y\}$ be a basis of B corresponding to the basis $\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\}$ of $Z_2(\mathbb{C})$. Then x and y satisfy the relations of example 2.3.1. Since $x^2 = 0$, there is an invertible $u \in M_2(\mathbb{C})$ such that $uxu^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The relations $xy = 0$ and $yx = x$ imply that $uyu^{-1} = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$ for some $b \in \mathbb{C}$. Further conjugating by $v = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ one obtains $vuxu^{-1}v^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $vuyu^{-1}v^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus the isomorphism between B and $Z_2(\mathbb{C})$ is realized as conjugation by vu.

We conclude that, up to conjugation and transposes, the only antisymmetric solutions of (AYB) in $M_2(\mathbb{C})$ are $r = 0$ and $r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

We finish this section with some general remarks on antisymmetric algebras and solutions of (AYB).

Any vector space carrying a non degenerate antisymmetric form must obviously be even dimensional, so the same is true for antisymmetric algebras (for instance, for the algebra of example 2.5, $\dim Z_m(B) = m(m-1) \dim B$ is an even number). More interestingly, antisymmetric algebras cannot be unital. In fact, if A is a unital algebra, then any Connes cocycle ω on A must be degenerate, since otherwise the bijective map $\tilde{\omega} : A \to A^*$, being a derivation by proposition 2.1, would vanish at 1. Regarding solutions of (AYB) for unital algebras, a stronger conclusion may in fact be derived:

Proposition 2.9. *Let $A \neq 0$ be a unital algebra. Then any solution $r \in A \otimes A$ of the associative Yang-Baxter equation, antisymmetric or not, is necessarily degenerate.*

Proof. Suppose there is a non degenerate solution r. In particular A is finite dimensional. By proposition 2.6, $\tilde{r} : A^* \to A$ is a morphism of ϵ-bialgebras. Hence A is self dual as ϵ-bialgebra, in particular it is both unital and counital. But this forces $A = 0$ by [A1, remark 2.2]. □

Remark 2.10. It is known that any antisymmetric solution of (CYB) in a semisimple Lie algebra must be degenerate [C-P, proposition 2.2.5]. There is an analogous result for semisimple ϵ-bialgebras. However, this becomes trivial in view of proposition 2.9, since it is well known that any semisimple associative algebra is necessarily unital.

3. From the associative to the classical Yang-Baxter equation

The main result we will prove in this section, theorem 3.5, states: *Suppose r^+ is A-invariant.*

1. *If $\mathbf{A}(r)=0$ then $\mathbf{C}(r)=0$.*
2. *If $\mathbf{A}(r)$ is A-invariant then $\mathbf{C}(r)$ is A^{lie}-invariant.*

Here, A^{lie} denotes the Lie algebra obtained by endowing A with the commutator bracket.

These are formal but not completely straightforward consequences of the definitions. Several results of independent interest will be derived in the course of the proof. The significance of theorem 3.5 is that a coboundary ϵ-bialgebra (A,r) for which r^+ is A-invariant yields a coboundary Lie bialgebra by passing to the commutator and cocommutator brackets (corollary 3.7).

Lemma 3.1. *For $s,t\in A\otimes A$, let $\{t,s\}=t_{13}s_{12}-t_{12}s_{23}+t_{23}s_{13}\in A\otimes A\otimes A$.*

1. *Let $s\in\Lambda^2(A)$ be an antisymmetric tensor and $t\in S^2(A)$ a symmetric one. Suppose that t is A-invariant. Then $\{t,s\}=-\{s,t\}$.*
2. *Let $r\in A\otimes A$ be such that r^+ is A-invariant. Then*

$$\mathbf{A}(r)=\mathbf{A}(r^+)+\mathbf{A}(r^-)\ .$$

Proof. 1. Write $t=\sum u_i\otimes v_i=\sum v_i\otimes u_i$ and $s=\sum x_j\otimes y_j=-\sum y_j\otimes x_j$. We compute

$$\begin{aligned}t_{13}s_{12}&=\sum u_ix_j\otimes y_j\otimes v_i=\sum v_ix_j\otimes y_j\otimes u_i=\sum v_i\otimes y_j\otimes x_ju_i\\&=\sum u_i\otimes y_j\otimes x_jv_i=-\sum u_i\otimes x_j\otimes y_jv_i=-s_{23}t_{13}\end{aligned}$$

by invariance, symmetry and antisymmetry. Invariance alone implies

$$\begin{aligned}t_{12}s_{23}&=\sum u_i\otimes v_ix_j\otimes y_j=\sum x_ju_i\otimes v_i\otimes y_j=s_{13}t_{12}\\t_{23}s_{13}&=\sum x_j\otimes u_i\otimes v_iy_j=\sum x_j\otimes y_ju_i\otimes v_i=s_{12}t_{23}\ .\end{aligned}$$

Thus

$$\{t,s\}=t_{13}s_{12}-t_{12}s_{23}+t_{23}s_{13}=-s_{23}t_{13}-s_{13}t_{12}+s_{12}t_{23}=-\{s,t\}\ .$$

2. Note that $\mathbf{A}(r)=\{r,r\}$. By part 1 applied to $t=r^+$ and $s=r^-$, $\{r^+,r^-\}=-\{r^-,r^+\}$. Hence

$$\mathbf{A}(r)=\{r^++r^-,r^++r^-\}=\{r^+,r^+\}+\{r^+,r^-\}+\{r^-,r^+\}+\{r^-,r^-\}=\mathbf{A}(r^+)+\mathbf{A}(r^-)\ .$$

□

Lemma 3.2. *Let $t\in A\otimes A$ be symmetric and A-invariant. Then t is also A^{op}-invariant.*

Proof. The A-bimodule and A^{op}-bimodule structures on the space $A\otimes A=A^{op}\otimes A^{op}$ are respectively

$$a\cdot(u\otimes v)\cdot b=au\otimes vb\text{ and }a^{op}\cdot(u\otimes v)\cdot b^{op}=ua\otimes bv\ .$$

If $t=\sum u_i\otimes v_i=\sum v_i\otimes u_i$ is symmetric then

$$a\cdot t\cdot b=\sum au_i\otimes v_ib=\sum av_i\otimes u_ib=\tau(\sum u_ib\otimes av_i)=\tau(b^{op}\cdot t\cdot a^{op})\ .$$

In particular,

$$a\cdot t-t\cdot a=\tau(t\cdot a^{op})-\tau(a^{op}\cdot t)=-\tau(a^{op}\cdot t-t\cdot a^{op})\ .$$

Hence t is A-invariant if and only if t is A^{op}-invariant. □

Lemma 3.3. *Let $r\in A\otimes A$ be A-invariant. Then*

$$\mathbf{A}(r)=r_{13}r_{12}=r_{12}r_{23}=r_{23}r_{13}$$

and this element of $A\otimes A\otimes A$ is A-invariant.

Proof. Write $r=\sum u_i\otimes v_i$. We compute

$$r_{13}r_{12}=\sum u_iu_j\otimes v_j\otimes v_i=\sum u_j\otimes v_ju_i\otimes v_i=r_{12}r_{23}$$

and

$$r_{23}r_{13}=\sum u_j\otimes u_i\otimes v_iv_j=\sum u_j\otimes v_ju_i\otimes v_i=r_{12}r_{23}$$

from which

$$r_{13}r_{12}=r_{12}r_{23}=r_{23}r_{13}=\mathbf{A}(r)\ .$$

The A-invariance of, say, $r_{13}r_{12}$, follows immediately from that of r. □

Suppose $r\in\Lambda^2(A)$ is antisymmetric. It is known that in this case $\mathbf{C}(r)$ is antisymmetric too, i.e. $\mathbf{C}(r)\in\Lambda^3(A)$. Next, we consider to what extent $\mathbf{A}(r)$ fails to be antisymmetric. To this end, consider the permutations

$$\sigma,\rho:A\otimes A\otimes A\to A\otimes A\otimes A,\ \sigma(x\otimes y\otimes z)=z\otimes y\otimes x \text{ and } \rho(x\otimes y\otimes z)=y\otimes z\otimes x\ .$$

We are interested in the more general situation where it is only assumed that the symmetric part of r is A-invariant (not necessarily zero). Consider the element

$$\mathbf{A}'(r)=r_{12}r_{13}-r_{23}r_{12}+r_{13}r_{23}\ .$$

Note that

$$\mathbf{C}(r)=\mathbf{A}(r)-\mathbf{A}'(r)\ . \tag{3.1}$$

Lemma 3.4. 1. *Let $s\in A\otimes A$ be antisymmetric and $t\in A\otimes A$ be symmetric. Then*

$$\sigma(\mathbf{A}(s))=\mathbf{A}'(s)\ ,\ \sigma(\mathbf{A}(t))=\mathbf{A}'(t) \text{ and } \rho(\mathbf{A}(s))=\mathbf{A}(s)\ .$$

2. *Let $r\in A\otimes A$ be such that r^+ is A-invariant. Then*

$$\sigma(\mathbf{A}(r))=\mathbf{A}'(r) \text{ and } \rho(\mathbf{A}(r))=\mathbf{A}(r)\ .$$

Proof. 1. We first prove the assertions involving σ. Write $s=\sum u_i\otimes v_i=-\sum v_i\otimes u_i$. We have

$$s_{13}s_{12}=\sum u_iu_j\otimes v_j\otimes v_i=-\sum u_iv_j\otimes u_j\otimes v_i=\sum v_iv_j\otimes u_j\otimes u_i$$
$$\Rightarrow\sigma(s_{13}s_{12})=\sum u_i\otimes u_j\otimes v_iv_j=s_{13}s_{23}\ .$$

Also,

$$s_{12}s_{23}=\sum u_j\otimes v_ju_i\otimes v_i=-\sum v_j\otimes u_ju_i\otimes v_i=\sum v_j\otimes u_jv_i\otimes u_i$$
$$\Rightarrow\sigma(s_{12}s_{23})=\sum u_i\otimes u_jv_i\otimes v_j=s_{23}s_{12}\ ;$$

and

$$s_{23}s_{13}=\sum u_j\otimes u_i\otimes v_iv_j=-\sum v_j\otimes u_i\otimes v_iu_j=\sum v_j\otimes v_i\otimes u_iu_j$$
$$\Rightarrow\sigma(s_{23}s_{13})=\sum u_iu_j\otimes v_i\otimes v_j=s_{12}s_{13}\ .$$

Thus

$$\sigma(\mathbf{A}(s))=\sigma(s_{13}s_{12}-s_{12}s_{23}+s_{23}s_{13})=s_{13}s_{23}-s_{23}s_{12}+s_{12}s_{13}=\mathbf{A}'(s)\ .$$

The same argument shows that $\sigma(\mathbf{A}(t))=\mathbf{A}'(t)$, since there were two sign changes involved at each step.

Finally,

$$\begin{aligned}\rho(\mathbf{A}(s)) &= \rho(\sum u_i u_j \otimes v_j \otimes v_i - \sum u_j \otimes v_j u_i \otimes v_i + \sum u_j \otimes u_i \otimes v_i v_j) \\ &= \sum v_j \otimes v_i \otimes u_i u_j - \sum v_j u_i \otimes v_i \otimes u_j + \sum u_i \otimes v_i v_j \otimes u_j \\ &= \sum u_j \otimes u_i \otimes v_i v_j + \sum u_j u_i \otimes v_i \otimes v_j - \sum u_i \otimes v_i u_j \otimes v_j = \mathbf{A}(s)\ .\end{aligned}$$

2. For the permutation σ, we apply the results just proved to $t = r^+$ and $s = r^-$, together with lemma 3.1.2, which applies since r^+ is A-invariant. We obtain

$$\sigma(\mathbf{A}(r)) = \sigma(\mathbf{A}(r^+)) + \sigma(\mathbf{A}(r^-)) = \mathbf{A}'(r^+) + \mathbf{A}'(r^-) = \mathbf{A}'(r)\ .$$

The last equality holds by lemma 3.1.2 applied to the algebra A^{op}: first, notice that r^+ is also A^{op}-invariant by lemma 3.2, so the lemma indeed applies; second, observe that $\mathbf{A}_{A^{op}}(r) = \mathbf{A}'_A(r)$.

For the permutation ρ, we need a special argument for the term $\mathbf{A}(r^+)$. Since r^+ is A-invariant, we can apply lemma 3.3:

$$\mathbf{A}(r^+) = r^+_{13}r^+_{12} = r^+_{23}r^+_{13}\ .$$

Writing $r^+ = \sum x_i \otimes y_i$, we have

$$\begin{aligned}\mathbf{A}(r^+) &= r^+_{13}r^+_{12} = \sum x_i x_j \otimes y_j \otimes y_i \\ \Rightarrow \rho(\mathbf{A}(r^+)) &= \sum y_j \otimes y_i \otimes x_i x_j = \sum x_j \otimes x_i \otimes y_i y_j = r^+_{23}r^+_{13} = \mathbf{A}(r^+)\ .\end{aligned}$$

Using this and the result of the previous part (for $s = r^-$) we conclude

$$\rho(\mathbf{A}(r)) = \rho(\mathbf{A}(r^+)) + \rho(\mathbf{A}(r^-)) = \mathbf{A}(r^+) + \mathbf{A}(r^-) = \mathbf{A}(r)\ .$$

□

Let A^{lie} denote the Lie algebra obtained from the associative algebra A via the commutator $[a,b] = ab - ba$. Given an A-bimodule M, let M^{lie} denote the same space M but viewed as an A^{lie}-module via

$$a \rightarrow m = a \cdot m - m \cdot a\ .$$

The space $M^{\otimes n}$ can then be seen as an A-bimodule via

$$a \cdot (m_1 \otimes m_2 \otimes \ldots \otimes m_n) \cdot b = am_1 \otimes m_2 \otimes \ldots \otimes m_n b\ ,$$

as the A^{lie}-module $(M^{\otimes n})^{lie}$

$$a \rightarrow (m_1 \otimes m_2 \otimes \ldots \otimes m_n) = am_1 \otimes m_2 \otimes \ldots \otimes m_n - m_1 \otimes m_2 \otimes \ldots \otimes m_n a\ ,$$

or as the A^{lie}-module $(M^{lie})^{\otimes n}$

$$\begin{aligned}&a \twoheadrightarrow (m_1 \otimes m_2 \otimes \ldots \otimes m_n) \\ &\quad = (a \rightarrow m_1) \otimes m_2 \otimes \ldots \otimes m_n + m_1 \otimes (a \rightarrow m_2) \otimes \ldots \otimes m_n + \cdots + m_1 \otimes m_2 \otimes \ldots \otimes (a \rightarrow m_n)\ .\end{aligned}$$

It is easy to see that these structures are related by means of the n-cycle $\rho_n : M^{\otimes n} \rightarrow M^{\otimes n}$,

$$\rho_n(m_1 \otimes m_2 \otimes \ldots \otimes m_n) = m_2 \otimes \ldots \otimes m_n \otimes m_1$$

as follows: for $a \in A$ and $u \in M^{\otimes n}$,

$$a \twoheadrightarrow u = a \rightarrow u + \rho_n^{-1}(a \rightarrow \rho_n(u)) + \cdots + \rho_n^{-(n-1)}(a \rightarrow \rho_n^{(n-1)}(u))\ . \tag{3.2}$$

We will consider A^{lie}-invariant elements of $M^{\otimes n}$ always with respect to the Lie structure $\twoheadrightarrow$. Thus, an element $m \in M^{\otimes n}$ is A-invariant if $a \rightarrow m = 0\ \forall\ a \in A$, while it is A^{lie}-invariant if $a \twoheadrightarrow m = 0\ \forall\ a \in A$.

We are mostly interested in the case $M = A$ and $n = 2$ or 3. Notice that the permutations $\rho_2 = \tau$ and $\rho_3 = \rho$ have been considered already.

We are now in position to prove the main result of this section.

Theorem 3.5. *Let $r \in A \otimes A$ be a tensor for which its symmetric part r^+ is A-invariant.*

1. *r^+ is then also A^{lie}-invariant.*
2. *If $\mathbf{A}(r) = 0$ then $\mathbf{C}(r) = 0$.*
3. *If $\mathbf{A}(r)$ is A-invariant then $\mathbf{C}(r)$ is A^{lie}-invariant.*

Proof. 1. Since $\tau(r^+) = r^+$, equation (3.2) gives

$$a \twoheadrightarrow r^+ = a \rightarrow r^+ + \tau(a \rightarrow r^+) = 0 \text{ for any } a \in A \ ,$$

as needed.

2. By lemma 3.4.2, $\mathbf{A}'(r) = \sigma(\mathbf{A}(r)) = 0$. Hence

$$\mathbf{C}(r) \overset{(3.1)}{=} \mathbf{A}(r) - \mathbf{A}'(r) = 0 \ .$$

3. By lemma 3.4.2, $\mathbf{A}(r) = \rho(\mathbf{A}(r)) = \rho^2(\mathbf{A}(r))$. By hypothesis this element is A-invariant. It follows from equation (3.2) that $\mathbf{A}(r)$ is A^{lie}-invariant.

Now, according to equation (3.1) and lemma 3.4.2,

$$\mathbf{C}(r) = \mathbf{A}(r) - \mathbf{A}'(r) = \mathbf{A}(r) - \sigma(\mathbf{A}(r)) \ .$$

Clearly, $\sigma : A^{lie} \otimes A^{lie} \otimes A^{lie} \rightarrow A^{lie} \otimes A^{lie} \otimes A^{lie}$ is a morphism of A^{lie}-modules (though not of A-bimodules). It follows that $\mathbf{C}(r)$ is A^{lie}-invariant.

□

Examples 3.6.

1. Consider the solution of (AYB) of example 2.3.3, in the algebra $Z_m(k)$ of matrices with last row equal to zero. Since it is antisymmetric, it is also a solution of (CYB) by theorem 3.5. Similarly, we may obtain a solution of (CYB) in the Lie algebra of matrices of size mn whose last n rows are zero, according to example 2.5. This solution of (CYB) was constructed by Belavin and Drinfeld [B-D, example in section 7]. The non degenerate Lie cocycle to which it corresponds already appears in [Oom, page 497]. The constructions of the present paper show that these solutions and cocycles in the Lie setting actually come from the associative setting.
2. Let A be a symmetric algebra (section 2) and $t \in A \otimes A$ the symmetric invariant tensor corresponding to the given form; such an element is often called a Casimir element. Then

$$r = \frac{t}{\mathbf{x} - \mathbf{y}}$$

is an antisymmetric solution of (AYB), belonging to a certain completed tensor product containing $(A \otimes A)[\mathbf{x}, \mathbf{y}]$ [A1, example 5.4.3]. In a sense, this may be seen as constructed from the solution $\frac{1}{\mathbf{x}-\mathbf{y}}$ in $k[\mathbf{x}]$ and the symmetric form on A, as in proposition 2.4.

Allowing ourselves to apply theorem 3.5 in this situation, part 2 says that r is also a solution of (CYB) in the loop Lie algebra $A^{lie}[\mathbf{x}]$. Notice that by part 1 of the theorem, t is also a Casimir element for the Lie algebra A^{lie}; therefore, this solution is the usual "rational" solution of Drinfeld [Dri, example 3.3] or [C-P, example 2.1.9].

We will now interpret the results of theorem 3.5 in terms of infinitesimal and Lie bialgebras.

Given $r = \sum u_i \otimes v_i \in A \otimes A$, let $\Delta_r : A \rightarrow A \otimes A$ and $\Delta_r^{lie} : A^{lie} \rightarrow A^{lie} \otimes A^{lie}$ be

$$\Delta_r(a) = a \rightarrow r = \sum a u_i \otimes v_i - \sum u_i \otimes v_i a$$

and

$$\Delta_r^{lie}(a) = a \twoheadrightarrow r = \sum [a, u_i] \otimes v_i + \sum u_i \otimes [a, v_i] \ .$$

Recall that the pair (A, r) is called a coboundary ϵ-bialgebra if (A, Δ_r) is an ϵ-bialgebra. According to [A1, proposition 5.1], this is the case if and only if $\mathbf{A}(r)$ is A-invariant. Similarly, (A^{lie}, r) is called a coboundary Lie bialgebra if $(A^{lie}, \Delta_r^{lie})$ is a Lie bialgebra. It is well known that this is the case if and only if $\mathbf{C}(r)$ and r^+ are A^{lie}-invariant [C-P, proposition 2.1.2]. Also, (A, r) is called a quasitriangular ϵ-bialgebra if $\mathbf{A}(r) = 0$ and (A^{lie}, r) is called a quasitriangular Lie bialgebra if $\mathbf{C}(r) = 0$ and r^+ is A^{lie}-invariant.

Corollary 3.7. *Let (A, r) be a coboundary ϵ-bialgebra for which the symmetric part r^+ is A-invariant.*

1. *Then (A^{lie}, r) is a coboundary Lie bialgebra. Moreover,*

$$\Delta_r^{lie} = \Delta_r - \tau\Delta_r \ .$$

2. *If in addition (A, r) is a quasitriangular ϵ-bialgebra then (A^{lie}, r) is a quasitriangular Lie bialgebra.*

Proof. If (A, r) is a coboundary ϵ-bialgebra then $\mathbf{A}(r)$ is A-invariant. Hence, by theorem 3.5.3, $\mathbf{C}(r)$ is A^{lie}-invariant. Also, by theorem 3.5.1, r^+ is A^{lie}-invariant. Thus (A^{lie}, r) is a Lie bialgebra. Moreover,

$$\Delta_r^{lie}(a) = a \twoheadrightarrow r \overset{(3.2)}{=} a \rightarrow r + \tau(a \rightarrow \tau(r)) = \Delta_r(a) + \tau\Delta_{\tau(r)}(a) \ ,$$

and since r^+ is A-invariant,

$$\Delta_r = \Delta_{r^-} \text{ and } \Delta_{\tau(r)} = -\Delta_{r^-} \ .$$

Hence

$$\Delta_r^{lie} = \Delta_{r^-} - \tau\Delta_{r^-} = \Delta_r - \tau\Delta_r \ .$$

The second assertion is just a restatement of theorem 3.5.2. □

Remark 3.8. For arbitrary ϵ-bialgebras (A, m, Δ), it is not true that $(A, m - m\tau, \Delta - \tau\Delta)$ is a Lie bialgebra (see example 4.12.1). However, the corollary shows that this does hold in the special case when $\Delta = \Delta_r$ and r^+ is A-invariant. More generally, we will show that if (A, m, Δ) is a *balanced* ϵ-bialgebra, then $(A, m - m\tau, \Delta - \tau\Delta)$ is indeed a Lie bialgebra (proposition 4.10). When $\Delta = \Delta_r$ and r^+ is A-invariant, (A, m, Δ_r) is balanced (proposition 4.7).

4. From infinitesimal to Lie bialgebras. Balanced infinitesimal bialgebras

Let (A, m, Δ) be an ϵ-bialgebra. The commutator bracket $m - m\tau$ and the cocommutator cobracket $\Delta - \tau\Delta$ endow A with structures of Lie algebra and coalgebra, but these are not always compatible: $(A, m - m\tau, \Delta - \tau\Delta)$ need not be a Lie bialgebra. In this section we introduce the class of *balanced* ϵ-bialgebras, for which this conclusion does hold. This class also arises naturally in the context of Drinfeld's double, as will be seen in section 5. Any ϵ-bialgebra that is both commutative and cocommutative is balanced (in this case the corresponding Lie bialgebra is trivial). A different family of balanced ϵ-bialgebras is provided by those coboundary ϵ-bialgebras (A, r) for which r^+ is A-invariant.

4.1. **The balanceator.** Balanced ϵ-bialgebras are defined below, in terms of a natural map $\mathbf{B} : A\otimes A \to A\otimes A$ that plays a role somewhat analogous to that of the commutator of an associative algebra and the cocommutator of a coassociative coalgebra, simultaneously. The basic properties of $\mathbf{B}$, as well the construction of a balanced ϵ-bialgebra from an arbitrary one (analogous to the construction of the center of an associative algebra), are also presented in this section.

Definition 4.1. Let (A, m, Δ) be an ϵ-bialgebra. The map $\mathbf{B} : A\otimes A \to A\otimes A$ defined by

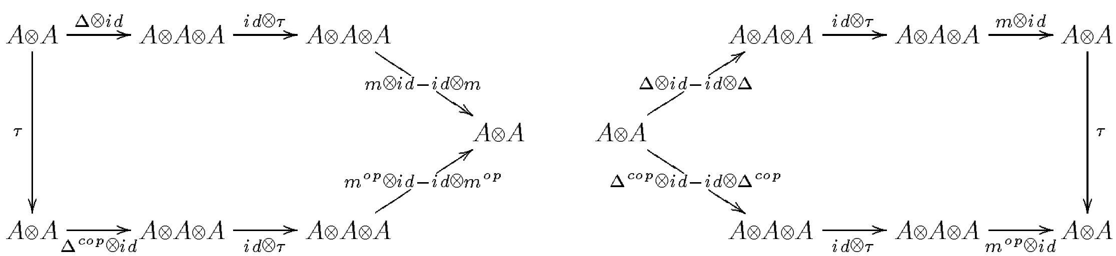
$$\mathbf{B}(a, b) = a \to \tau\Delta(b) + \tau\Big(b \to \tau\Delta(a)\Big) = ab_2\otimes b_1 - b_2\otimes b_1 a + a_1\otimes ba_2 - a_1 b\otimes a_2$$

is called the *balanceator* of A. The ϵ-bialgebra A is called *balanced* if $\mathbf{B} \equiv 0$, or equivalently if for every a, $b \in A$,

$$a_1 b\otimes a_2 + b_2\otimes b_1 a = ab_2\otimes b_1 + a_1\otimes ba_2 \ . \tag{B}$$

The relevance of these notions will hopefully become clear from the results of this section.

Remark 4.2. One may easily check that condition (B) is equivalent to the commutativity of any of the diagrams below:



Since these diagrams are clearly dual to each other, condition (B) is self dual.

Recall from [A1, section 2] that if $A := (A, m, \Delta)$ is an ϵ-bialgebra then so are $A^{op,cop} := (A, m^{op}, \Delta^{cop})$, $A^{-,+} := (A, -m, \Delta)$, $A^{+,-} := (A, m, -\Delta)$ and, if A is finite dimensional, $A^* := (A^*, \Delta^*, m^*)$.

Proposition 4.3. *The balanceator* $\mathbf{B}$ *enjoys the following properties.*

1. $\mathbf{B}$ *is natural: for any morphism* $f : A \to B$ *of* ϵ*-bialgebras, diagram* $\begin{array}{ccc} A\otimes A & \xrightarrow{\mathbf{B}_A} & A\otimes A \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f\otimes f} \\ B\otimes B & \xrightarrow[\mathbf{B}_B]{} & B\otimes B \end{array}$ *commutes.*
2. *Diagram* $\begin{array}{ccc} A\otimes A & \xrightarrow{\mathbf{B}} & A\otimes A \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ A\otimes A & \xrightarrow[\mathbf{B}]{} & A\otimes A \end{array}$ *commutes.*
3. *For any* ϵ*-bialgebra* A,
$$\mathbf{B}_{A^{op,cop}} = -\tau\mathbf{B}_A, \quad \mathbf{B}_{A^*} = (\mathbf{B}_A)^* \quad \textit{and} \quad \mathbf{B}_{A^{-,+}} = \mathbf{B}_{A^{+,-}} = -\mathbf{B}_A \ .$$
Hence, if A *is balanced, so are the other associated* ϵ*-bialgebras.*
4. *For any* $a, b, c \in A$,
$$\mathbf{B}(ab, c) = (a\otimes 1)\mathbf{B}(b, c) + \mathbf{B}(a, c)(1\otimes b) \ , \tag{4.1}$$
$$\mathbf{B}(a, bc) = \mathbf{B}(a, b)(c\otimes 1) + (1\otimes b)\mathbf{B}(a, c) \ . \tag{4.2}$$
5. *For any* $a, b \in A$,
$$(\Delta\otimes id)\mathbf{B}(a, b) = a_1\otimes\mathbf{B}(a_2, b) + (id\otimes\tau)\Big(\mathbf{B}(a, b_1)\otimes b_2\Big) \ , \tag{4.3}$$
$$(id\otimes\Delta)\mathbf{B}(a, b) = \mathbf{B}(a_1, b)\otimes a_2 + (\tau\otimes id)\Big(b_1\otimes\mathbf{B}(a, b_2)\Big) \ . \tag{4.4}$$

Proof. Parts 1 and 2 are immediate from the definition. Part 3 can be checked by direct computation or by expressing $\mathbf{B}$ in terms of the diagrams of remark 4.2, and noting that they are dual to each other, and that they remain unchanged after replacing m for m^{op} and Δ for Δ^{cop}.

To verify (4.1), recall that by (1.1), $\Delta(ab) = ab_1 \otimes b_2 + a_1 \otimes a_2 b$, hence

$$\begin{aligned}\mathbf{B}(ab,c) &= abc_2 \otimes c_1 - c_2 \otimes c_1 ab + ab_1 \otimes cb_2 + a_1 \otimes ca_2 b - ab_1 c \otimes b_2 - a_1 c \otimes a_2 b \\ &= abc_2 \otimes c_1 + ab_1 \otimes cb_2 - ab_1 c \otimes b_2 + (-ac_2 \otimes c_1 b + ac_2 \otimes c_1 b) - c_2 \otimes c_1 ab + a_1 \otimes ca_2 b - a_1 c \otimes a_2 b \\ &= (a \otimes 1)\mathbf{B}(b,c) + \mathbf{B}(a,c)(1 \otimes b) \ .\end{aligned}$$

Equation (4.2) follows from (4.1) and part 2. Equations (4.3) and (4.4) are their formal duals, so they must also hold according to remark 4.2; a direct verification is also possible. □

Remark 4.4. Equations (4.1) and (4.2) involve a unit element 1, but it is of course not necessary to assume that it belongs to A, since we may always embed A in an a unital algebra. If A does have a unit element 1, then it follows from these equations that

$$\mathbf{B}(1,a) = \mathbf{B}(a,1) = 0 \ \forall \ a \in A \ .$$

Equations (4.1) and (4.2) are reminiscent of the familiar properties of the commutator bracket of an associative algebra, e.g. $[ab,c] = a[b,c] + [a,c]b$. Also, (4.3) and (4.4) are similar to the dual properties of cocommutators. This suggests that condition (B) may be seen as a simultaneous weakening of commutativity and cocommutativity. This is further supported by the results of propositions 4.5 and 4.6 below.

Proposition 4.5. *Let (A,m,Δ) be an ϵ-bialgebra that is both commutative and cocommutative. Then A is balanced.*

Proof. We verify that condition (B) holds, as follows:

$$a_1 b \otimes a_2 + b_2 \otimes b_1 a = ba_1 \otimes a_2 + b_1 \otimes b_2 a \overset{(1.1)}{=} \Delta(ba) = \Delta(ab) \overset{(1.1)}{=} ab_1 \otimes b_2 + a_1 \otimes a_2 b = ab_2 \otimes b_1 + a_1 \otimes ba_2 \ .$$

□

The analogy between the balanceator of an ϵ-bialgebra and the commutator of an algebra suggests the possibility of defining for ϵ-bialgebras a notion analogous to that of the center of an algebra. It is pleasant that this indeed results in a balanced ϵ-subbialgebra of the given ϵ-bialgebra. Moreover, since the balanceator is self dual, a construction of a balanced quotient of an arbitrary ϵ-bialgebra, dual to the previous one, is also possible.

Proposition 4.6. *Let A be an arbitrary ϵ-bialgebra.*

1. *Define*

$$Z_b(A) = \{a \in A \ / \ \mathbf{B}(a,b) = 0 \ \forall \ b \in A\} = \{b \in A \ / \ \mathbf{B}(a,b) = 0 \ \forall \ a \in A\} \ .$$

Then $Z_b(A)$ is an ϵ-subbialgebra of A, and as such, it is balanced.

2. *Define*

$$I_b(A) = \sum_{f \in A^*} \mathsf{Im}\Big((id \otimes f)\mathbf{B}\Big) = \sum_{f \in A^*} \mathsf{Im}\Big((f \otimes id)\mathbf{B}\Big) \ .$$

Then $I_b(A)$ is a biideal of A, and the quotient $A/I_b(A)$ is a balanced ϵ-bialgebra.

Proof. First notice that the two definitions of $Z_b(A)$ indeed agree, by symmetry of $\mathbf{B}$ (proposition 4.3.2). Equation (4.1) immediately implies that $Z_b(A)$ is a subalgebra. Notice that $Z_b(A) = \cap_{b \in A} \mathsf{Ker}(\mathbf{B}(-,b))$; therefore, to show that it is a subcoalgebra we must show that

$$\Delta\Big(Z_b(A)\Big) \subseteq Z_b(A) \otimes Z_b(A) = \Big(Z_b(A) \otimes A\Big) \cap \Big(A \otimes Z_b(A)\Big) = \bigcap_{b \in A} \mathsf{Ker}\Big(\mathbf{B}(-,b) \otimes id\Big) \cap \mathsf{Ker}\Big(id \otimes \mathbf{B}(-,b)\Big)$$

Take $a \in Z_b(A)$, then

$$\Big(\mathbf{B}(-,b)\otimes id\Big)\Delta(a) = \mathbf{B}(a_1,b)\otimes a_2 \overset{(4.4)}{=} (id\otimes\Delta)\mathbf{B}(a,b) - (\tau\otimes id)\Big(b_1\otimes\mathbf{B}(a,b_2)\Big) = 0$$

$$\Big(id\otimes\mathbf{B}(-,b)\Big)\Delta(a) = a_1\otimes\mathbf{B}(a_2,b) \overset{(4.3)}{=} (\Delta\otimes id)\mathbf{B}(a,b) - (id\otimes\tau)\Big(\mathbf{B}(a,b_1)\otimes b_2\Big) = 0$$

as needed. Thus, $Z_b(A)$ is a subbialgebra of A. It is obviously balanced since, by naturality of $\mathbf{B}$, $\mathbf{B}_{Z_b(A)}(a,b) = \mathbf{B}_A(a,b) = 0 \ \forall \ a,b \in Z_b(A)$.

Part 2 is formally dual to part 1; nevertheless, we provide a direct verification.

The elements of the form $(id\otimes f)\mathbf{B}(b,c)$ are linear generators of $I_b(A)$. Identity (4.1) implies that

$$a\cdot(id\otimes f)\mathbf{B}(b,c) = (id\otimes f)\mathbf{B}(ab,c) - \Big(id\otimes(b\rightarrow f)\Big)\mathbf{B}(a,c) \in I_b(A) \ ,$$

where $b \rightarrow f$ is the usual left action of A on A^*, see (5.10). Thus, $I_b(A)$ is a left ideal. Similarly, one deduces from (4.2) that it is a right ideal. Finally, (4.3) implies that

$$\Delta(id\otimes f)\mathbf{B}(a,b) = a_1\otimes(id\otimes f)\mathbf{B}(a_2,b) + (id\otimes f)\mathbf{B}(a,b_1)\otimes b_2 \in A\otimes I_b(A) + I_b(A)\otimes A$$

which shows that $I_b(A)$ is a coideal. Finally, to show that $A/I_b(A)$ is balanced one may use the following general fact: if S is a subspace of $A\otimes A$, $S_1 = \sum_{f\in A^*}(f\otimes id)(S)$ and $S_2 = \sum_{f\in A^*}(id\otimes f)(S)$, then $S \subseteq S_2\otimes S_1$. Applied to $S = \mathsf{Im}(\mathbf{B})$ this gives $\mathsf{Im}(\mathbf{B}) \subseteq I_b(A)\otimes I_b(A)$, which certainly implies that $\mathbf{B}_{A/I_b(A)} \equiv 0$. □

Examples of balanced ϵ-bialgebras and of the constructions of proposition 4.6 will be given below (examples 4.8 and 4.12). In proposition 5.10, the balanceator of the Drinfeld double will be explicitly calculated

Balanced quasitriangular ϵ-bialgebras. Below we find a sufficient condition for a quasitriangular (or more generally, coboundary) ϵ-bialgebra to be balanced. This provides an important source of examples of balanced ϵ-bialgebras.

Proposition 4.7. *Let (A,r) be a coboundary ϵ-bialgebra. Then*

$$\mathbf{B}(a,b) = 2\cdot(a\rightarrow\tau\Delta_{r^+}(b)) \ \forall \ a, \ b\in A \ .$$

In particular, if r^+ is A-invariant then A is balanced.

Proof. Let $r = \sum u_i\otimes v_i$, so $\Delta_r(a) = \sum au_i\otimes v_i - u_i\otimes v_i a$. We compute

$$\begin{aligned} a\rightarrow\tau\Delta_r(b) &= a\rightarrow\sum(v_i\otimes bu_i - v_ib\otimes u_i) = a\rightarrow\Big((1\otimes b)\tau(r) - \tau(r)(b\otimes 1)\Big) \\ &= (a\otimes b)\tau(r) - (a\otimes 1)\tau(r)(b\otimes 1) - (1\otimes b)\tau(r)(1\otimes a) + \tau(r)(b\otimes a) \ . \end{aligned}$$

Hence

$$b\rightarrow\tau\Delta_r(a) = (b\otimes a)\tau(r) - (b\otimes 1)\tau(r)(a\otimes 1) - (1\otimes a)\tau(r)(1\otimes b) + \tau(r)(a\otimes b)$$

and

$$\begin{aligned} \tau\Big(b\rightarrow\tau\Delta_r(a)\Big) &= (a\otimes b)r - (1\otimes b)r(1\otimes a) - (a\otimes 1)r(b\otimes 1) + r(b\otimes a) \\ &= a\rightarrow\tau\Delta_{\tau(r)}(b) \ . \end{aligned}$$

Hence

$$\mathbf{B}(a,b) = a\rightarrow\tau\Delta_r(b) + \tau\Big(b\rightarrow\tau\Delta_r(a)\Big) = a\rightarrow\tau\Delta_{r+\tau(r)}(b) = 2\cdot(a\rightarrow\tau\Delta_{r^+}(b)) \ .$$

□

Examples 4.8.

1. The ϵ-bialgebra $k[\varepsilon]/(\varepsilon^2)$ of dual numbers is quasitriangular with $r = 1\otimes\varepsilon$ [A1, example 5.4.1]. Since $r^+ = 1\otimes\varepsilon + \varepsilon\otimes 1$ is A-invariant, this ϵ-bialgebra is balanced. (Actually, it is commutative and cocommutative.)
2. The condition of proposition 4.7 is not necessary: a coboundary ϵ-bialgebra (A, r) may be balanced even if r^+ is not invariant. Consider the three dimensional algebra $A = k[\varepsilon_1, \varepsilon_2]$ where
$$\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = 0 \ .$$
A can be realized as the subalgebra of $M_3(k)$ of matrices of the form $\begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & c & a \end{bmatrix}$.

   A is quasitriangular with $r = 1\otimes\varepsilon_1$ [A1, example 5.4.1]. We have
$$\Delta_{r^+}(1) = 0,\ \Delta_{r^+}(\varepsilon_1) = 0 \text{ but } \Delta_{r^+}(\varepsilon_2) = \frac{1}{2}(\varepsilon_2\otimes\varepsilon_1 - \varepsilon_1\otimes\varepsilon_2) \neq 0 \ .$$
In particular r^+ is not A-invariant. However,
$$1 \to \Delta_{r^+}(\varepsilon_2) = 0,\ \varepsilon_1 \to \Delta_{r^+}(\varepsilon_2) = 0 \text{ and } \varepsilon_2 \to \Delta_{r^+}(\varepsilon_2) = 0 \ .$$
Therefore, according to proposition 4.7, $\mathbf{B}(a,b) = 2\cdot(a \to \tau\Delta_{r^+}(b)) = 0\ \forall\ a,\ b \in A$, so A is balanced.

From balanced ϵ-bialgebras to Lie bialgebras. Below we will show that balanced ϵ-bialgebras give rise to Lie bialgebras by passing to the commutator and cocommutator brackets. In fact, the necessary and sufficient condition for obtaining a Lie bialgebra is that the balanceator satisfies $\mathbf{B}(a,b) = \mathbf{B}(b,a)$. Several examples will be given.

Lemma 4.9. *Let M be an A-bimodule and M^{lie} the associated A^{lie}-module: $a \to m = a\cdot m - m\cdot a$. Then, $\mathsf{Der}(A, M) \subseteq \mathsf{Der}(A^{lie}, M^{lie})$.*

Proof. If $D \in \mathsf{Der}(A, M)$ then
$$D([a,b]) = D(ab) - D(ba) = a\cdot D(b) - D(a)\cdot b - \Big(b\cdot D(a) - D(b)\cdot a\Big) = a \to D(b) - b \to D(a)$$
so $D \in \mathsf{Der}(A^{lie}, M^{lie})$. □

Proposition 4.10. *Let (A, m, Δ) be an ϵ-bialgebra. Then $(A, m - m\tau, \Delta - \tau\Delta)$ is a Lie bialgebra if and only if $\mathbf{B} = \mathbf{B}\tau$. In particular, if (A, m, Δ) is balanced ($\mathbf{B} = 0$) then $(A, m - m\tau, \Delta - \tau\Delta)$ is a Lie bialgebra.*

Proof. By hypothesis and lemma 4.9,
$$\Delta \in \mathsf{Der}(A, A\otimes A) \subseteq \mathsf{Der}(A^{lie}, (A\otimes A)^{lie}) \ .$$
Thus,
$$\Delta([a,b]) = a \to \Delta(b) - b \to \Delta(a)$$
and, letting $\Delta^{lie} = \Delta - \tau\Delta$,
$$(*)\qquad \Delta^{lie}([a,b]) = a \to \Delta(b) - b \to \Delta(a) - \tau\Big(a \to \Delta(b) - b \to \Delta(a)\Big) \ .$$
On the other hand, we have
$$a \twoheadrightarrow \Delta^{lie}(b) = a \twoheadrightarrow \Delta(b) - a \twoheadrightarrow \tau\Delta(b) \overset{(3.2)}{=} a \to \Delta(b) + \tau(a \to \tau\Delta(b)) - a \to \tau\Delta(b) - \tau(a \to \Delta(b)) \ .$$

Therefore

$$\begin{aligned}
&a \twoheadrightarrow \Delta^{lie}(b) - b \twoheadrightarrow \Delta^{lie}(a)\\
&= a \to \Delta(b) - b \to \Delta(a) + \tau\Big(a \to \tau\Delta(b) - b \to \tau\Delta(a)\Big) - a \to \tau\Delta(b) + b \to \tau\Delta(a) - \tau\Big(a \to \Delta(b) - b \to \Delta(a)\Big)\\
&\overset{(*)}{=} \Delta^{lie}([a,b]) + \tau\Big(a \to \tau\Delta(b) - b \to \tau\Delta(a)\Big) - a \to \tau\Delta(b) + b \to \tau\Delta(a)\\
&= \Delta^{lie}([a,b]) + \mathbf{B}(b,a) - \mathbf{B}(a,b)\ .
\end{aligned}$$

Thus, $\Delta^{lie} \in \mathsf{Der}(A^{lie}, A^{lie}\otimes A^{lie})$ if and only if $\mathbf{B}(b,a) = \mathbf{B}(a,b)$ for every $a,\, b \in A$, which is the desired conclusion. □

Remarks 4.11.

1. Recall that $\mathbf{B}\tau = \tau\mathbf{B}$ always holds, by proposition 4.3. Thus, when the condition $\mathbf{B} = \mathbf{B}\tau$ holds, the balanceator induces a map $S^2(A) \to S^2(A)$, and conversely. We will say in this case that the balanceator is symmetric.
2. It is easy to see the balanceator is symmetric if and only if $a \to \tau\Delta(b) - b \to \tau\Delta(a)$ is a symmetric element of $A\otimes A$, for every $a, b \in A$.

Examples 4.12.

1. Not every ϵ-bialgebra yields a Lie bialgebra. To see this, we must produce an ϵ-bialgebra for which $\mathbf{B}$ is not symmetric. Consider the algebra of matrices $M_2(k)$ equipped with the comultiplication

$$\Delta \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & z \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix}\ .$$

$M_2(k)$ is then an ϵ-bialgebra [A1, example 2.3.7]. One calculates

$$\mathbf{B}(\begin{bmatrix} x & y \\ z & w \end{bmatrix}, \begin{bmatrix} p & q \\ r & s \end{bmatrix}) = 2\cdot(\begin{bmatrix} 0 & x \\ 0 & z \end{bmatrix} \otimes \begin{bmatrix} 0 & p \\ 0 & r \end{bmatrix} - \begin{bmatrix} xr & xs \\ zr & zs \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} pz & pw \\ rz & rw \end{bmatrix} + \begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix})\ .$$

Thus $\mathbf{B}$ is not symmetric, so there is no associated Lie bialgebra.

It is easy to see that the constructions of proposition 4.6 produce the following results: $Z_b(A)$ is the subalgebra of matrices of the form $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ (where the comultiplication is zero), while $A/I_b(A) = 0$.

2. This is an example of a non balanced ϵ-bialgebra for which the balanceator is symmetric (and therefore there is an associated Lie bialgebra). Consider the algebra $A = k[\mathbf{x}]/(\mathbf{x}^4)$, equipped with the comultiplication

$$\Delta(\mathbf{x}^i) = \mathbf{x}^i \otimes \mathbf{x}^2 - 1\otimes\mathbf{x}^{i+2}\ .$$

A is then an ϵ-bialgebra. In fact, if $r = 1\otimes\mathbf{x}^2$ then (A, r) is quasitriangular and $\Delta = \Delta_r$ [A1, example 5.4.1].

In this case

$$\begin{aligned}
\mathbf{B}(\mathbf{x}^i, \mathbf{x}^j) = {} & \mathbf{x}^{i+2}\otimes\mathbf{x}^j - \mathbf{x}^{i+j+2}\otimes 1 - \mathbf{x}^2\otimes\mathbf{x}^{i+j} + \mathbf{x}^{j+2}\otimes\mathbf{x}^i\\
& + \mathbf{x}^j\otimes\mathbf{x}^{i+2} - 1\otimes\mathbf{x}^{i+j+2} - \mathbf{x}^{i+j}\otimes\mathbf{x}^2 + \mathbf{x}^i\otimes\mathbf{x}^{j+2}\ ,
\end{aligned}$$

which is clearly symmetric (this had to be the case since (A, m) is commutative, so $(A, m - m\tau, \Delta - \tau\Delta)$ is a Lie bialgebra trivially). However, $\mathbf{B} \neq 0$ so A is not balanced.

The constructions of proposition 4.6 are as follows: $Z_b(A)$ is the subalgebra spanned by 1 and $\mathbf{x}^2$, which is isomorphic to the ϵ-bialgebra of dual numbers (example 4.8.1); $I_b(A)$ is the ideal generated by $\mathbf{x}$, so $A/I_b(A) \cong k$ with zero comultiplication.

3. Let $\mathfrak{g}$ be the two dimensional non abelian Lie algebra: $\mathfrak{g}$ has a basis $\{x, y\}$ such that $[y, x] = x$. There are, up to isomorphism and scalar multiples, only two Lie bialgebra structures on $\mathfrak{g}$ [C-P, examples 1.3.7 and 2.1.5]:

$$\delta_1(x) = 0, \quad \delta_1(y) = y \otimes x - x \otimes y$$

and

$$\delta_2(x) = x \otimes y - y \otimes x, \quad \delta_2(y) = 0 \ .$$

Let us discuss these examples in connection with balanced ϵ-bialgebras.

Let A be the two dimensional algebra with basis $\{x, y\}$ and structure

$$x^2 = 0, \ xy = 0, \ yx = x \text{ and } y^2 = y \ ,$$
$$\Delta(x) = x \otimes x \text{ and } \Delta(y) = y \otimes x \ .$$

A is then an ϵ-bialgebra; in fact, it is the quasitriangular ϵ-bialgebra of example 2.3.1, with $r = y \otimes x - x \otimes y$. In this case, $r^+ = 0$, so by propositions 4.7 and 4.10, A is balanced and A^{lie} is a Lie bialgebra. This is precisely the first Lie bialgebra structure on $\mathfrak{g}$ above.

Direct calculations show that the second Lie bialgebra structure does not come from any ϵ-bialgebra structure.

5. Drinfeld's double of Lie and infinitesimal bialgebras

Recall that the Drinfeld double $D(\mathfrak{g})$ of a Lie bialgebra $\mathfrak{g}$ carries a non degenerate symmetric form that is associative, in the sense that

$$\langle [\alpha, \beta], \gamma \rangle = \langle \alpha, [\beta, \gamma] \rangle \ \forall \ \alpha, \beta, \gamma \in D(\mathfrak{g}) \ .$$

In [A1], we have defined the Drinfeld double $D(A)$ of an ϵ-bialgebra A. Since this is defined for arbitrary ϵ-bialgebras A (balanced or not), there may be no Lie bialgebras associated to A or $D(A)$. $D(A)$ always carries a canonical symmetric associative form, but this may be degenerate. The main result of this section describes a necessary and sufficient condition for A to be balanced in terms of the radical of this form (theorem 5.5). In this case, the quotient $D_b(A)$ of $D(A)$ by the radical of the form is again a balanced ϵ-bialgebra (proposition 5.10), and the corresponding Lie bialgebra is, up to a sign, the Drinfeld double of the Lie bialgebra corresponding to A (proposition 5.12). Another important result in this section is theorem 5.9, which contains the universal property of $D_b(A)$.

We begin by recalling the construction of the double $D(A)$ of a finite dimensional ϵ-bialgebra (A, m, Δ). Consider the following version of the dual of A

$$A' := (A^*, \Delta^{*^{op}}, -m^{*^{cop}}) \ .$$

Explicitly, the structure on A' is:

$$(f \cdot g)(a) = g(a_1)f(a_2) \ \forall \ a \in A, \ f, g \in A' \text{ and} \tag{5.1}$$

$$\Delta(f) = f_1 \otimes f_2 \iff f(ab) = -f_2(a)f_1(b) \ \forall \ f \in A', \ a, b \in A \ . \tag{5.2}$$

Below we always refer to this structure when dealing with multiplications or comultiplications of elements of A'. Consider also the actions of A' on A and A on A' defined by

$$f \rightarrow a = f(a_1)a_2 \text{ and } f \leftarrow a = -f_2(a)f_1 \tag{5.3}$$

or equivalently

$$g(f \rightarrow a) = (gf)(a) \text{ and } (f \leftarrow a)(b) = f(ab) \ . \tag{5.4}$$

Proposition 5.1. *Let A be a finite dimensional ϵ-bialgebra, consider the vector space*

$$D(A) := (A \otimes A') \oplus A \oplus A'$$

and denote the element $a \otimes f \in A \otimes A' \subseteq D(A)$ by $a \bowtie f$. Then $D(A)$ admits a unique ϵ-bialgebra structure such that:

(a) *A and A' are subalgebras, $a \cdot f = a \bowtie f$, $f \cdot a = f \rightarrow a + f \leftarrow a$ and*
(b) *A and A' are subcoalgebras.*

Explicitly, the multiplication and comultiplication are given in the remaining cases by

$$a \cdot (b \bowtie f) = ab \bowtie f \ , \quad (a \bowtie f) \cdot g = a \bowtie fg \ ,$$

$$(a \bowtie f) \cdot b = a(f \rightarrow b) + a \bowtie (f \leftarrow b) \ , \quad f \cdot (a \bowtie g) = (f \rightarrow a) \bowtie g + (f \leftarrow a)g \ ,$$

$$(a \bowtie f) \cdot (b \bowtie g) = a(f \rightarrow b) \bowtie g + a \bowtie (f \leftarrow b)g \ , \tag{5.5}$$

$$\Delta(a \bowtie f) = (a \bowtie f_1) \otimes f_2 + a_1 \otimes (a_2 \bowtie f) \ . \tag{5.6}$$

Proof. See [A1, theorem 7.3]. □

If $\mathfrak{g}$ is a finite dimensional Lie bialgebra, its double $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ carries a non degenerate symmetric associative form, defined by

$$\langle a + f, b + g \rangle = f(b) + g(a) \ .$$

For ϵ-bialgebras, the analogous form is still symmetric and associative, in the sense that

$$\langle \alpha\beta, \gamma \rangle = \langle \alpha, \beta\gamma \rangle \ \forall \ \alpha, \beta, \gamma \in D(A) \ ,$$

but this form is usually degenerate.

Proposition 5.2. *Let A be a finite dimensional ϵ-bialgebra. There is a symmetric associative form on $D(A)$ uniquely determined by*

$$\langle a + f, b + g \rangle = f(b) + g(a) \ . \tag{5.7}$$

Proof. Define a bilinear form on $D(A)$ by

$$\begin{aligned} \langle a + f + b \bowtie g, a' + f' + b' \bowtie g' \rangle =& f'(a) + f(a') + g'(ab') + g(a'b) + (g'f)(b') + (gf')(b) \\ &+ g(b'_1)g'(bb'_2) + g'(b_1)g(b'b_2) \ . \end{aligned} \tag{5.8}$$

This form is clearly symmetric and satisfies equation 5.7. Conversely, symmetry, associativity and 5.7 force us to define the form as above: for instance,

$$\langle b \bowtie g, a' \rangle = \langle a', b \bowtie g \rangle = \langle a', b \cdot g \rangle = \langle a' \cdot b, g \rangle \overset{(5.7)}{=} g(a'b) \ .$$

Thus, uniqueness holds. To show that this form is indeed associative one must check several cases. We do it only for one of the most relevant, involving elements of $A \otimes A' \subseteq D(A)$. In the computation, we will use that

$$\Delta\Big(a(f \rightarrow b)\Big) \overset{(5.3)}{=} f(b_1)\Delta(ab_2) \overset{(1.1)}{=} f(b_1)ab_2 \otimes b_3 + f(b_1)a_1 \otimes a_2 b_2 \ . \tag{$*$}$$

We calculate

$$\begin{aligned}\langle (a \bowtie f)\cdot(b\bowtie g), c\bowtie h\rangle &\overset{(5.5)}{=} \langle a(f\to b)\bowtie g + a\bowtie (f\leftarrow b)g, c\bowtie h\rangle\\ &\overset{(5.8),\,(*)}{=} g(c_1)h\Big(a(f\to b)c_2\Big) + h\Big(f(b_1)ab_2\Big)g(cb_3) + h\Big(f(b_1)a_1\Big)g(ca_2b_2)\\ &\qquad + \Big((f\leftarrow b)g\Big)(c_1)h(ac_2) + h(a_1)\Big((f\leftarrow b)g\Big)(ca_2)\\ &\overset{(5.3),\,(5.4)}{=} \underbrace{g(c_1)f(b_1)h(ab_2c_2)}_{1} + \underbrace{f(b_1)h(ab_2)g(cb_3)}_{2} + \underbrace{f(b_1)h(a_1)g(ca_2b_2)}_{3}\\ &\qquad + \underbrace{g(c_1)f(bc_2)h(ac_3)}_{4} + \underbrace{h(a_1)g(ca_2)f(ba_3)}_{5} + \underbrace{h(a_1)g(c_1)f(bc_2a_2)}_{6}\ .\end{aligned}$$

Since the form is symmetric, we can also conclude that

$$\begin{aligned}\langle a\bowtie f, (b\bowtie g)\cdot(c\bowtie h)\rangle &= \langle (b\bowtie g)\cdot(c\bowtie h), a\bowtie f\rangle\\ &= \underbrace{h(a_1)g(c_1)f(bc_2a_2)}_{6} + \underbrace{g(c_1)f(bc_2)h(ac_3)}_{4} + \underbrace{g(c_1)f(b_1)h(ab_2c_2)}_{1}\\ &\qquad + \underbrace{h(a_1)g(ca_2)f(ba_3)}_{5} + \underbrace{f(b_1)h(ab_2)g(cb_3)}_{2} + \underbrace{f(b_1)h(a_1)g(ca_2b_2)}_{3}\ .\end{aligned}$$

Thus

$$\langle (a\bowtie f)\cdot(b\bowtie g), c\bowtie h\rangle = \langle a\bowtie f, (b\bowtie g)\cdot(c\bowtie h)\rangle$$

as needed. □

We will refer to the form of proposition 5.2 as the *canonical* form on the double. We now turn towards the proof of the main theorem of this section, which relates the radical of the canonical form with a certain coideal of the double, to be defined next.

Let A be a finite dimensional ϵ-bialgebra. There are four natural actions of A and A' on each other. Our definition of $D(A)$ involves only two of them, namely the actions (5.3) and (5.4). It is time now to consider the other two actions, defined by

$$a\leftarrow f = f(a_2)a_1 \text{ and } a\to f = -f_1(a)f_2 \tag{5.9}$$

or equivalently

$$g(a\leftarrow f) = (fg)(a) \text{ and } (a\to f)(b) = f(ba)\ . \tag{5.10}$$

Let I denote the subspace of $D(A)$ linearly spanned by the set

$$\{a\bowtie f - a\leftarrow f - a\to f \ /\ a\in A,\ f\in A'\}\ . \tag{5.11}$$

Recall that in $D(A)$ we have (by proposition 5.1)

$$f\cdot a = f\to a + f\leftarrow a \quad\text{and}\quad a\cdot f = a\bowtie f\ ;$$

therefore, modulo I, the multiplication in $D(A)$ acquires the more symmetric form

$$f\cdot a \equiv f\to a + f\leftarrow a \quad\text{and}\quad a\cdot f \equiv a\leftarrow f + a\to f\ .$$

We will return to this after discussing the basic properties of I.

Lemma 5.3. *Let A be a finite dimensional ϵ-bialgebra.*

1. *For any $a\in A$ and $f\in A'$,*

$$(a\leftarrow f_1)\otimes f_2 + a_1\otimes(a_2\to f) = 0\ . \tag{5.12}$$

2. *I is a coideal of* $D(A)$.

Proof. Evaluating the left hand side on $b \in A$ one obtains

$$f_2(b)(a \leftarrow f_1) + (a_2 \rightarrow f)(b)a_1 \overset{(5.9),(5.10)}{=} f_2(b)f_1(a_2)a_1 + f(ba_2)a_1 \overset{(5.1)}{=} 0$$

which proves the first part.

For the second, we calculate

$$\begin{aligned}
&\Delta(a \bowtie f - a \leftarrow f - a \rightarrow f)\\
&\overset{(5.9)}{=} \Delta(a \bowtie f) - f(a_2)\Delta(a_1) + f_1(a)\Delta(f_2) \overset{(5.6)}{=} (a \bowtie f_1)\otimes f_2 + a_1\otimes(a_2 \bowtie f) - f(a_3)a_1\otimes a_2 + f_1(a)f_2\otimes f_3\\
&\overset{(5.9)}{=} (a \bowtie f_1)\otimes f_2 + a_1\otimes(a_2 \bowtie f) - a_1\otimes(a_2 \leftarrow f) - (a \rightarrow f_1)\otimes f_2\\
&\overset{(5.12)}{=} (a \bowtie f_1 - a \rightarrow f_1 - a \leftarrow f_1)\otimes f_2 + a_1\otimes(a_2 \bowtie f - a_2 \leftarrow f - a_2 \rightarrow f)\ .
\end{aligned}$$

Thus,

$$\Delta(I) \subseteq I\otimes D(A) + D(A)\otimes I$$

as needed. □

Lemma 5.4. *As vector spaces,* $D(A) \cong I \oplus A \oplus A'$.

Proof. Consider the linear map $\pi : D(A) \rightarrow A \oplus A'$ given by

$$\pi(a) = a,\ \pi(f) = f \text{ and } \pi(a \bowtie f) = a \leftarrow f + a \rightarrow f\ .$$

By definition, $I \subseteq \mathsf{Ker}(\pi)$. Conversely, if $\alpha = \sum a_i \bowtie f_i + a + f \in \mathsf{Ker}(\pi)$ then $\sum(a_i \leftarrow f_i + a_i \rightarrow f_i) + a + f = 0$, hence $\alpha = \sum a_i \bowtie f_i - \sum a_i \leftarrow f_i - \sum a_i \rightarrow f_i \in I$. Thus $I = \mathsf{Ker}(\pi)$.

Let $i : A\oplus A' \rightarrow D(A)$ be the inclusion. Then $\pi i = id$, hence $D(A) \cong \mathsf{Ker}(\pi)\oplus\mathsf{Im}(i) = I\oplus A\oplus A'$. □

We are now in position to derive the main result of this section, which characterizes balanced ϵ-bialgebras in terms of the radical of the canonical form on the double and the coideal I.

Theorem 5.5. *Let A be a finite dimensional ϵ-bialgebra and I the subspace linearly spanned by*

$$\{a \bowtie f - a \leftarrow f - a \rightarrow f\ /\ a \in A,\ f \in A'\}$$

(as above). The following statements are equivalent:

1. *A is balanced;*
2. *I is the radical of the canonical form on* $D(A)$*;*
3. *I is an ideal of* $D(A)$.

Proof.
<u>1 ⇒ 2.</u> We first show that $I \subseteq \mathsf{rad}\langle\ ,\ \rangle$. For this we consider three cases:

$$(i)\ \langle a \bowtie f - a \leftarrow f - a \rightarrow f, b\rangle \overset{(5.8)}{=} f(ba) - (a \rightarrow f)(b) \overset{(5.9)}{=} f(ba) - f(ba) = 0;$$

$$(ii)\ \langle a \bowtie f - a \leftarrow f - a \rightarrow f, g\rangle \overset{(5.8)}{=} (fg)(a) - g(a \leftarrow f) \overset{(5.10)}{=} (fg)(a) - (fg)(a) = 0;$$

$$\begin{aligned}
(iii)\ \langle a \bowtie f - a \leftarrow f - a \rightarrow f, b \bowtie g\rangle &\overset{(5.8)}{=} f(b_1)g(ab_2) + g(a_1)f(ba_2) - g\Big((a \leftarrow f)b\Big) - \Big(g(a \rightarrow f)\Big)(b)\\
&\overset{(5.9),(5.10)}{=} f(b_1)g(ab_2) + g(a_1)f(ba_2) - f(a_2)g(a_1b) - g(b_2)f(b_1a) = 0;
\end{aligned}$$

the last equality holds precisely by condition (B) (definition 4.1).

Since $I \subseteq \mathsf{rad}\langle\ ,\ \rangle$, the form $\langle\ ,\ \rangle$ descends to a bilinear form on $D(A)/I$ with radical $\mathsf{rad}\langle\ ,\ \rangle/I$. By lemma 5.4, $D(A)/I \cong A \oplus A'$, and the resulting form is simply $\langle a+f, b+g\rangle = f(b)+g(a)$, which is clearly non degenerate. Hence $\mathsf{rad}\langle\ ,\ \rangle = I$.

$\underline{2 \Rightarrow 3.}$ This is clear since, by proposition 5.2, the canonical form is associative.

$\underline{3 \Rightarrow 1.}$ Since I is an ideal, we have in particular that

$$(a \bowtie f - a \leftarrow f - a \rightarrow f)b \equiv 0 \mod I \quad \forall\ a,b \in A,\ f \in A' \ .$$

Now, according to proposition 5.1,

$$\begin{aligned}(a \bowtie f - a \leftarrow f - a \rightarrow f)b &= a(f \rightarrow b) + a \bowtie (f \leftarrow b) - (a \leftarrow f)b - (a \rightarrow f) \rightarrow b - (a \rightarrow f) \leftarrow b \\ &= f(b_1)ab_2 + a \bowtie (f \leftarrow b) - f(a_2)a_1b - f(b_1a)b_2 - a \rightarrow (f \leftarrow b)\ ,\end{aligned}$$

by equations (5.3), (5.4), (5.9) and (5.10). Hence, modulo I, we have

$$\begin{aligned}0 \equiv& f(b_1)ab_2 + a \leftarrow (f \leftarrow b) - f(a_2)a_1b - f(b_1a)b_2 \mod I \\ 0 \equiv& f(b_1)ab_2 + f(ba_2)a_1 - f(a_2)a_1b - f(b_1a)b_2 \mod I\end{aligned}$$

Since $I \cap A = 0$ (by lemma 5.4), we deduce that

$$0 = f(b_1)ab_2 + f(ba_2)a_1 - f(a_2)a_1b - f(b_1a)b_2$$

and since this holds $\forall\ f \in A^*$, we conclude that

$$0 = b_1 \otimes ab_2 + ba_2 \otimes a_1 - a_2 \otimes a_1b - b_1a \otimes b_2\ ,$$

which is condition (B). □

Let A be a finite dimensional balanced ϵ-bialgebra. It follows from theorem 5.5 and lemma 5.3 that $D(A)/I$ is an ϵ-bialgebra. According to lemma 5.4, the inclusion $A \oplus A' \hookrightarrow D(A)$ induces an isomorphism of vector spaces $D(A)/I \cong A \oplus A'$. We denote by $D_b(A)$ the resulting ϵ-bialgebra structure on $A \oplus A'$, and call it the *balanced Drinfeld double* of A. From the definition of I, this structure is as follows. The algebra structure is determined by

(a) A and A' are subalgebras,
(b) $a \cdot f = a \leftarrow f + a \rightarrow f$ and
(c) $f \cdot a = f \rightarrow a + f \leftarrow a$.

Equivalently, and writing (a,f) instead of $a+f$,

$$(a,f) \cdot (b,g) = (ab + a \leftarrow g + f \rightarrow b,\ a \rightarrow g + f \leftarrow b + fg)\ .$$

As a coalgebra, $D_b(A)$ is simply the direct sum of A and A'.

Theorem 5.5 also shows that $D_b(A)$ is a symmetric algebra. The non degenerate symmetric associative form is simply

$$\langle a+f, b+g\rangle = f(b) + g(a)\ .$$

Example 5.6. The ϵ-bialgebra $k[\varepsilon]/(\varepsilon^2)$ of dual numbers (example 4.8.1) is the simplest instance of a balanced Drinfeld double. In fact, if we view the one dimensional algebra k as a balanced ϵ-bialgebra with comultiplication $\Delta(1) = 0$, then $D_b(k)$ is two dimensional, with basis $\{1, \varepsilon\}$, where $\{1\}$ and $\{\varepsilon\}$ are dual bases of k and k'. The algebra structure on the balanced double is precisely that of the algebra of dual numbers, while the coalgebra structure has been multiplied by -1 with respect to that of example 4.8.1.

We will derive next a universal property of $D_b(A)$ as a quasitriangular ϵ-bialgebra. First, we recall the corresponding result for the full double $D(A)$ and state a lemma needed for the proof.

Lemma 5.7. *Let A be a finite dimensional ϵ-bialgebra, $\{e_i\}$ be a linear basis of A and $\{f_i\}$ the dual basis of A'. Then $\forall\ a \in A$ and $f \in A'$,*

$$\sum_i f_i(a)e_i = a \text{ and } \sum_i f(e_i)f_i = f \tag{5.13}$$

$$\sum_i ae_i \otimes f_i = \sum_i e_i \otimes (f_i \leftarrow a) \tag{5.14}$$

$$\sum_i (f \rightarrow e_i) \otimes f_i = \sum_i e_i \otimes f_i f \tag{5.15}$$

$$\sum_i e_i a \otimes f_i = \sum_i e_i \otimes (a \rightarrow f_i) \tag{5.16}$$

$$\sum_i (e_i \leftarrow f) \otimes f_i = \sum_i e_i \otimes f f_i \tag{5.17}$$

$$\sum_i e_i \otimes (f_i \rightarrow a) = \sum_i (a \leftarrow f_i) \otimes e_i \tag{5.18}$$

$$\sum_i f_i \otimes (f \leftarrow e_i) = \sum_i (e_i \rightarrow f) \otimes f_i \tag{5.19}$$

Proof. These are all straightforward. A proof of (5.13) to (5.15) can be found in [A1, lemma 7.2]; the others are similar. □

Proposition 5.8. *Let A be a finite dimensional ϵ-bialgebra and $D(A)$ its Drinfeld double. Let $\{e_i\}$ and $\{f_i\}$ be dual bases of A and A' and*

$$R = -\sum_i e_i \otimes f_i \in A \otimes A' \subseteq D(A) \otimes D(A) \ .$$

1. *$(D(A), R)$ is a quasitriangular ϵ-bialgebra.*
2. *Let $r = \sum_j u_j \otimes v_j$ be an element of $A \otimes A$ and $\pi^r : D(A) \rightarrow A$ the map defined by*

$$\pi^r(a) = a,\ \pi^r(f) = -\sum_j f(u_j)v_j \text{ and } \pi^r(a \bowtie f) = -\sum_j f(u_j)av_j \ .$$

   Then (A, r) is quasitriangular if and only if π^r is a morphism of ϵ-bialgebras.
3. *A is quasitriangular if and only if the canonical inclusion $A \hookrightarrow D(A)$ splits as a morphism of ϵ-bialgebras.*

Proof. See [A1, theorem 7.3 and proposition 7.5]. □

The balanced Drinfeld double satisfies a similar universal property, but among quasitriangular ϵ-bialgebras (A, r) for which the symmetric part of r is A-invariant.

Theorem 5.9. *Let A be a finite dimensional balanced ϵ-bialgebra and*

$$R_b = -\sum_i e_i \otimes f_i \in A \otimes A' \subseteq D_b(A) \otimes D_b(A) \ .$$

1. *$(D_b(A), R_b)$ is a quasitriangular ϵ-bialgebra and R_b^+ is $D_b(A)$-invariant.*
2. *Let $r = \sum_j u_j \otimes v_j$ be an element of $A \otimes A$ and $\pi_b^r : D_b(A) \rightarrow A$ the map defined by*

$$\pi_b^r(a) = a \text{ and } \pi_b^r(f) = -\sum_j f(u_j)v_j \ .$$

   Then π_b^r is a morphism of ϵ-bialgebras if and only if (A, r) is quasitriangular and r^+ is A-invariant.

3. *The canonical inclusion* $A \hookrightarrow D_b(A)$ *splits as a morphism of* ϵ*-bialgebras if and only if there is* $r \in A \otimes A$ *such that* (A, r) *is quasitriangular and* r^+ *is* A*-invariant.*

Proof.

1. Since $(D(A), R)$ is a quasitriangular ϵ-bialgebra, so is its quotient $(D_b(A), R_b)$. To show that R_b^+ is $D_b(A)$-invariant we must check that $a \cdot R_b^+ = R_b^+ \cdot a$ and $f \cdot R_b^+ = R_b^+ \cdot f \ \forall\ a \in A,\ f \in A'$. Using the description of $D_b(A)$ following theorem 5.5, we compute

$$\begin{aligned} a \cdot 2R_b^+ &= a \cdot R_b + a \cdot \tau(R_b) = -\sum (a \cdot e_i) \otimes f_i - \sum (a \cdot f_i) \otimes e_i \\ &= -\underbrace{\sum a e_i \otimes f_i}_{A} - \underbrace{\sum (a \leftarrow f_i) \otimes e_i}_{B} - \underbrace{\sum (a \rightarrow f_i) \otimes e_i}_{C} \end{aligned}$$

On the other hand,

$$\begin{aligned} 2R_b^+ \cdot a &= R_b \cdot a + \tau(R_b) \cdot a = -\sum e_i \otimes (f_i \cdot a) - \sum f_i \otimes (e_i \cdot a) \\ &= -\underbrace{\sum e_i \otimes (f_i \rightarrow a)}_{B'} - \underbrace{\sum e_i \otimes (f_i \leftarrow a)}_{A'} - \underbrace{\sum f_i \otimes e_i a}_{C'} \ . \end{aligned}$$

Note that $A = A'$ by (5.14), $B = B'$ by (5.18) and $C = C'$ by (5.16). Thus $a \cdot R_b^+ = R_b^+ \cdot a$.
   Similarly,

$$\begin{aligned} f \cdot 2R_b^+ &= f \cdot R_b + f \cdot \tau(R_b) = -\sum (f \cdot e_i) \otimes f_i - \sum (f \cdot f_i) \otimes e_i \\ &= -\underbrace{\sum (f \rightarrow e_i) \otimes f_i}_{A} - \underbrace{\sum (f \leftarrow e_i) \otimes f_i}_{B} - \underbrace{\sum f f_i \otimes e_i}_{C} \end{aligned}$$

and

$$\begin{aligned} 2R_b^+ \cdot f &= R_b \cdot f + \tau(R_b) \cdot f = -\sum e_i \otimes (f_i \cdot f) - \sum f_i \otimes (e_i \cdot f) \\ &= -\underbrace{\sum e_i \otimes f_i f}_{A'} - \underbrace{\sum f_i \otimes (e_i \leftarrow f)}_{C'} - \underbrace{\sum f_i \otimes (e_i \rightarrow f)}_{B'} \ . \end{aligned}$$

We have that $A = A'$ by (5.15), $B = B'$ by (5.19) and $C = C'$ by (5.17). Thus $f \cdot R_b^+ = R_b^+ \cdot f$. This completes the proof of part 1.

2. The map π_r^b is surjective. Moreover,

$$(\pi_r^b \otimes \pi_r^b)(R_b) = -\sum_i \pi_r^b(e_i) \otimes \pi_r^b(f_i) = \sum_{i,j} f_i(u_j) e_i \otimes v_j \overset{(5.13)}{=} \sum_j u_j \otimes v_j = r \ .$$

Therefore, if π_r^b is a morphism of ϵ-bialgebras, (A, r) is a quotient ϵ-bialgebra of $(D_b(A), R_b)$, so it is quasitriangular and r^+ is A-invariant.
   Conversely, if (A, r) is quasitriangular then the map $\pi_r : D(A) \to A$ of proposition 5.8 is a morphism of ϵ-bialgebras. We claim that

$$\pi_r(I) = 0 \iff r^+ \text{ is } A\text{-invariant.}$$

This will complete the proof of part 2, since the map induced by π_r on the quotient $D(A)/I = D_b(A)$ is clearly π_r^b. To verify the claim, we calculate

$$\begin{aligned}\pi_r^b(a \bowtie f - a \leftarrow f - a \rightarrow f) &= -\sum_j f(u_j)av_j - a \leftarrow f - \sum_j (a \rightarrow f)(u_j)v_j \\ &= -\sum_j f(u_j)av_j - f(a_2)a_1 + \sum_j f(u_j a)v_j \ .\end{aligned}$$

Since (A, r) is quasitriangular,

$$a_1 \otimes a_2 = \Delta_r(a) = \sum_j au_j \otimes v_j - \sum_j u_j \otimes v_j a \Rightarrow f(a_2)a_1 = \sum_j f(v_j)au_j - \sum_j f(v_j a)u_j \ .$$

Therefore,

$$\begin{aligned}\pi_r(I) = 0 &\iff -\sum_j f(u_j)av_j - \sum_j f(v_j)au_j + \sum_j f(v_j a)u_j + \sum_j f(u_j a)v_j = 0 \ \ \forall\, a \in A,\ f \in A' \\ &\iff \sum_j av_j \otimes u_j + \sum_j au_j \otimes v_j = \sum_j u_j \otimes v_j a + \sum_j v_j \otimes u_j a \ \ \forall\, a \in A \\ &\iff a \cdot r^+ = r^+ \cdot a \ \ \forall\, a \in A \\ &\iff r^+ \text{ is } A\text{-invariant.}\end{aligned}$$

3. If (A, r) is quasitriangular and r^+ is A-invariant then the inclusion $A \hookrightarrow D_b(A)$ is split by the morphism π_b^r, according to part 2.

   Conversely, if the inclusion $A \hookrightarrow D_b(A)$ splits as a morphism of ϵ-bialgebras, then by part 1, $r := (\pi \otimes \pi)(R_b) \in A \otimes A$ is such that (A, r) is quasitriangular and r^+ is invariant, where $\pi : D_b(A) \rightarrow A$ is a splitting.

□

The properties of $D_b(A)$ thus obtained are strictly analogous to those of the double $D(\mathfrak{g})$ of a Lie bialgebra $\mathfrak{g}$. This analogy can be made more precise: we will show below that the functor from balanced ϵ-bialgebras to Lie bialgebras commutes with the double constructions. First we must verify that $D_b(A)$ is indeed a balanced ϵ-bialgebra. This follows at once from theorem 5.9 and proposition 4.7. However, we find of independent interest to deduce this result from the explicit expression of the balanceator of the full double $D(A)$, as follows.

Proposition 5.10. *Let A be a finite dimensional ϵ-bialgebra, $a \in A$ and $f \in A'$. Then the balanceator of the full double $D(A)$ is given by*

$$\mathbf{B}_{D(A)}(a, f) = (a \bowtie f_2 - a \rightarrow f_2 - a \leftarrow f_2) \otimes f_1 + (a_1 \rightarrow f + a_1 \leftarrow f - a_1 \bowtie f) \otimes a_2 \tag{5.20}$$

$$\mathbf{B}_{D(A)}(f, a) = f_1 \otimes (a \bowtie f_2 - a \rightarrow f_2 - a \leftarrow f_2) + a_2 \otimes (a_1 \rightarrow f + a_1 \leftarrow f - a_1 \bowtie f) \tag{5.21}$$

In particular, if A is balanced, then so is $D_b(A)$.

Proof. We have

$$\begin{aligned}\mathbf{B}_{D(A)}(a, f) =& af_2 \otimes f_1 - f_2 \otimes f_1 a + a_1 \otimes fa_2 - a_1 f \otimes a_2 \\ \overset{5.1}{=}& (a \bowtie f_2) \otimes f_1 - f_2 \otimes (f_1 \rightarrow a) - f_2 \otimes (f_1 \leftarrow a) + a_1 \otimes (f \rightarrow a_2) + a_1 \otimes (f \leftarrow a_2) - (a_1 \bowtie f) \otimes a_2 \\ \overset{(5.3)}{=}& (a \bowtie f_2) \otimes f_1 - f_1(a_1) f_2 \otimes a_2 + f_2(a) f_3 \otimes f_1 + f(a_2) a_1 \otimes a_3 - f_2(a_2) a_1 \otimes f_1 - (a_1 \bowtie f) \otimes a_2 \\ =& (a \bowtie f_2 + f_2(a) f_3 - f_2(a_2) a_1) \otimes f_1 - f_1(a_1) f_2 \otimes a_2 + f(a_2) a_1 \otimes a_3 - (a_1 \bowtie f) \otimes a_2 \\ \overset{(5.9)}{=}& (a \bowtie f_2 - a \rightarrow f_2 - a \leftarrow f_2) \otimes f_1 + (a_1 \rightarrow f + a_1 \leftarrow f - a_1 \bowtie f) \otimes a_2 \ .\end{aligned}$$

Hence also

$$\mathbf{B}_{D(A)}(f,a)\overset{4.3.2}{=}\tau\mathbf{B}_{D(A)}(a,f)=f_1\otimes(a\bowtie f_2-a\to f_2-a\leftarrow f_2)+a_2\otimes(a_1\to f+a_1\leftarrow f-a_1\bowtie f)\ .$$

Now suppose that A is balanced. Then so is A' by proposition 4.3.3. Hence, since A and A' are ϵ-subbialgebras of $D(A)$, and by naturality of $\mathbf{B}$,

$$\mathbf{B}_{D(A)}(a,b)=\mathbf{B}_{D(A)}(f,g)=0\ \ \forall\ a,b\in A,\ f,g\in A'\ .$$

On the other hand, by definition of I (5.11), equations (5.20) and (5.21) imply that

$$\mathbf{B}_{D(A)}(a,f)\subseteq I\otimes D(A)\text{ and }\mathbf{B}_{D(A)}(f,a)\subseteq D(A)\otimes I\ \ \forall\ a\in A,\ f\in A'\ .$$

Since A is balanced, I is an ideal of $D(A)$, by theorem 5.5. Together with the multiplicativity properties of $\mathbf{B}$ (equations (4.1) and (4.2)), and since $D(A)$ is generated as an algebra by A and A', these facts ensure that

$$\mathbf{B}_{D(A)}(\alpha,\beta)\in I\otimes D(A)+D(A)\otimes I\ \ \forall\ \alpha,\beta\in D(A)\ ,$$

which in turn says that

$$\mathbf{B}_{D_b(A)}\equiv 0\ ,$$

i.e. $D_b(A)$ is balanced. □

Remark 5.11. In proposition 4.6 we gave a canonical construction of a balanced quotient $A/I_b(A)$ of an arbitrary ϵ-bialgebra A. The expression for the balanceator of the full double suggests a connection between $D(A)/I_b(D(A))$ and $D_b(A)$.

Claim: Let A be a balanced finite dimensional ϵ-bialgebra. If A is unital or counital, there is a surjection

$$D_b(A)\twoheadrightarrow D(A)/I_b(D(A))\ .$$

Proof: Suppose that A is unital. Let $F\in D(A)^*$ be given by $F(a)=0$ and $F(f)=-f(1)\ \forall\ a\in A$, $f\in A'$. Since $f_1(1)f_2=-f$ (by (5.2)), we have that

$$(id\otimes F)\mathbf{B}_{D(A)}(a,f)\overset{(5.20)}{=}a\bowtie f-a\to f-a\leftarrow f\ \ \forall\ a\in A,\ f\in A'\ .$$

This shows that $I\subseteq I_b(D(A))$ and hence $D_b(A)=D(A)/I\twoheadrightarrow D(A)/I_b(D(A))$. When A is counital, the same conclusion can be obtained similarly.

In general, there is no connection between $D(A)/I_b(D(A))$ and $D_b(A)$. Consider for instance the trivial case when both the multiplication and comultiplication of A are zero. From (5.11) we see that $I=A\otimes A'\subseteq D(A)$. On the other hand, it follows from (5.20) and (5.21) that $\mathbf{B}_{D(A)}\equiv 0$, so $I_b(D(A))=0$.

For our final result, we must recall the construction of the Drinfeld double of a finite dimensional Lie bialgebra $(\mathfrak{g},[\ ,\ ],\delta)$. First, the dual Lie bialgebra $\mathfrak{g}^*$ has Lie bracket and cobracket defined by

$$[f,g](x):=(f\otimes g)\delta(x)\quad\text{and}\quad\delta(f)=f_1\otimes f_2\text{ iff }f[x,y]=f_1(x)f_2(y)\ \forall\ x,y\in\mathfrak{g}\ .$$

The *right coadjoint* actions of $\mathfrak{g}$ and $\mathfrak{g}^*$ on each other are defined by

$$(f\twoheadleftarrow x)(y):=f[x,y]\quad\text{and}\quad g(x\twoheadleftarrow f):=[f,g](x)\ .$$

The Drinfeld double of $\mathfrak{g}$ is the vector space $D(\mathfrak{g})=\mathfrak{g}\oplus\mathfrak{g}^*$ with the following Lie bracket and cobracket:

$$[(x,f),(y,g)]:=([x,y]+x\twoheadleftarrow g-y\twoheadleftarrow f,\ -g\twoheadleftarrow x+f\twoheadleftarrow y+[f,g])\tag{5.22}$$

and

$$\delta(x,f):=\delta(x)-\delta(f)\ .\tag{5.23}$$

Let $\{e_i\}$ and $\{f_i\}$ be dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$. It is well known that $(D(\mathfrak{g}), \sum e_i \otimes f_i)$ is a quasitriangular Lie bialgebra [E-S, sections 4.1 and 4.2].

According to proposition 4.10, if $A := (A, m, \Delta)$ is a balanced ϵ-bialgebra, then $A^{lie} := (A, m - m\tau, \Delta - \tau\Delta)$ is a Lie bialgebra. Since $D_b(A)$ is also balanced, it has an associated Lie bialgebra. As our last result, we verify that this is, up to isomorphism, the Drinfeld double of the Lie bialgebra associated to A.

Proposition 5.12. *Let A be a finite dimensional ϵ-bialgebra. Then the map*

$$D(A^{lie}) \to D_b(A)^{lie}, \quad (a, f) \to (a, -f)$$

is an isomorphism of quasitriangular Lie bialgebras.

Proof. We only need to check that the map preserves brackets and cobrackets. From the description of $D_b(A)$ following theorem 5.5 it follows that the (commutator) bracket on $D_b(A)^{lie}$ is

$$\begin{aligned}&[(a, f), (b, g)]\\ &\qquad = (ab - ba + a \leftarrow g - g \rightarrow a + f \rightarrow b - b \leftarrow f,\ a \rightarrow g - g \leftarrow a + f \leftarrow b - b \rightarrow f + fg - gf).\end{aligned}$$

Comparing definitions we find that

$$ab - ba = [a, b] \ ;$$

$$\begin{aligned}(fg - gf)(a) \overset{(5.1)}{=} g(a_1)f(a_2) - f(a_1)g(a_2) = -(f \otimes g)\delta_{A^{lie}}(a) = -[f, g]_{A^{lie}}(a)&\\ \Rightarrow fg - gf = -[f, g]_{A^{lie}} \ ;&\end{aligned}$$

$$\begin{aligned}f(a \leftarrow g - g \rightarrow a) \overset{(5.10), (5.4)}{=} (gf - fg)(a) = -[g, f]_{A^{lie}}(a)&\\ \Rightarrow a \leftarrow g - g \rightarrow a = -a \twoheadleftarrow g \text{ and hence also } f \rightarrow b - b \leftarrow f = b \twoheadleftarrow f \ ;&\end{aligned}$$

$$\begin{aligned}(a \rightarrow g - g \leftarrow a)(b) \overset{(5.10), (5.4)}{=} g(ba) - g(ab) = -g[a, b]&\\ \Rightarrow a \rightarrow g - g \leftarrow a = -g \twoheadleftarrow a \text{ and hence also } f \leftarrow b - b \rightarrow f = f \twoheadleftarrow b \ .&\end{aligned}$$

Thus, the Lie bracket on $D_b(A)^{lie}$ is

$$[(a, f), (b, g)] = ([a, b] - a \twoheadleftarrow g + b \twoheadleftarrow f,\ -g \twoheadleftarrow a + f \twoheadleftarrow b - [f, g]) \ .$$

Under the map $(a, f) \mapsto (a, -f)$, this does correspond to the Lie bracket on $D(A^{lie})$ given by equation (5.22).

On the other hand, the comultiplication on $D_b(A)$ is

$$\Delta(a, f) = \Delta_A(a) + \Delta_{A'}(f) = \Delta_A(a) - \tau\Delta_{A^*}(f) \ ,$$

hence the cobracket on $D_b(A)^{lie}$ is

$$\begin{aligned}\Delta^{lie}(a, f) &= \Delta(a, f) - \tau\Delta(a, f) = \Delta_A(a) - \tau\Delta_{A^*}(f) - \tau\Delta_A(a) + \Delta_{A^*}(f)\\ &= \delta_{A^{lie}}(a) + \delta_{(A^*)^{lie}}(f) \ .\end{aligned}$$

Under the map $(a, f) \mapsto (a, -f)$, this does correspond to the cobracket on $D(A^{lie})$,

$$\delta(a, f) \overset{(5.23)}{=} \delta_{A^{lie}}(a) - \delta_{(A^*)^{lie}}(f) \ .$$

Finally, recall that by corollary 3.7, $(D_b(A)^{lie}, -\sum e_i \otimes f_i)$ is a quasitriangular Lie bialgebra. This obviously corresponds to the quasitriangular structure on $D(A^{lie})$ under the isomorphism above. ☐

One may summarize the result of proposition 5.12 by means of the diagram below, which commutes up to isomorphism (or up to sign)

$$\begin{array}{ccc} \{\text{balanced } \epsilon\text{-bialgebras}\} & \xrightarrow{D_b} & \{\text{balanced } \epsilon\text{-bialgebras}\} \\ {\scriptstyle (\ )^{lie}}\downarrow & & \downarrow{\scriptstyle (\ )^{lie}} \\ \{\text{Lie bialgebras}\} & \xrightarrow[D]{} & \{\text{Lie bialgebras}\} \end{array}$$

The change of sign is not really essential; it may be avoided by making a different choice of signs in the definition of $D(A)$ (and hence of $D_b(A)$).

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