INFINITESIMAL BIALGEBRAS, PRE-LIE AND DENDRIFORM ALGEBRAS

MARCELO AGUIAR

Abstract. We introduce the categories of infinitesimal Hopf modules and bimodules over an infinitesimal bialgebra. We show that they correspond to modules and bimodules over the infinitesimal version of the double. We show that there is a natural, but non-obvious way to construct a pre-Lie algebra from an arbitrary infinitesimal bialgebra and a dendriform algebra from a quasitriangular infinitesimal bialgebra. As consequences, we obtain a pre-Lie structure on the space of paths on an arbitrary quiver, and a striking dendriform structure on the space of endomorphisms of an arbitrary infinitesimal bialgebra, which combines the convolution and composition products. We extend the previous constructions to the categories of Hopf, pre-Lie and dendriform bimodules. We construct a brace algebra structure from an arbitrary infinitesimal bialgebra; this refines the pre-Lie algebra construction. In two appendices, we show that infinitesimal bialgebras are comonoid objects in a certain monoidal category and discuss a related construction for counital infinitesimal bialgebras.

1. Introduction

The main results of this paper establish connections between infinitesimal bialgebras, pre-Lie algebras and dendriform algebras, which were a priori unexpected.

An infinitesimal bialgebra (abbreviated ϵ-bialgebra) is a triple (A, μ, Δ) where (A, μ) is an associative algebra, (A, Δ) is a coassociative coalgebra, and Δ is a derivation (see Section 2). We write $\Delta(a) = a_1 \otimes a_2$, omitting the sum symbol.

Infinitesimal bialgebras were introduced by Joni and Rota [17, Section XII]. The basic theory of these objects was developed in [1, 3], where analogies with the theories of ordinary Hopf algebras and Lie bialgebras were found; among which we remark the existence of a "double" construction analogous to that of Drinfeld for ordinary Hopf algebras or Lie bialgebras. On the other hand, infinitesimal bialgebras have found important applications in combinatorics [4, 11].

A pre-Lie algebra is a vector space P equipped with an operation $x \circ y$ satisfying a certain axiom (3.1), which guarantees that $x \circ y - y \circ x$ defines a Lie algebra structure on P. These objects were introduced by Gerstenhaber [13], whose terminology we follow, and independently by Vinberg [29]. See [8, 7] for more references, examples, and some of the general theory of pre-Lie algebras.

We show that any ϵ-bialgebra can be turned into a pre-Lie algebra by defining

$$a \circ b = b_1 a b_2 \,.$$

This is Theorem 3.2. As an application, we construct a canonical pre-Lie structure on the space of paths on an arbitrary quiver. We also note that the Witt Lie algebra arises in this way from the ϵ-bialgebra of divided differences (Examples 3.4). Other properties of this construction are provided in Section 3.

A dendriform algebra is a space D equipped with two operations $x \succ y$ and $x \prec y$ satisfying certain axioms (4.1), which guarantee that $x \succ y + x \prec y$ defines an associative algebra structure on D.

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Dendriform algebras were introduced by Loday [20, Chapter 5]. See [6, 26, 21, 22] for additional recent work on this subject.

There is a special class of ϵ-bialgebras for which the derivation Δ is principal, called quasitriangular ϵ-bialgebras. These are defined from solutions $r = \sum u_i \otimes v_i$ of the associative Yang-Baxter equation, introduced in [1] and reviewed in Section 2 of this paper. In Theorem 4.6, we show that any quasitriangular ϵ-bialgebra can be made into a dendriform algebra by defining

$$x \succ y = \sum_i u_i x v_i y \quad \text{and} \quad x \prec y = \sum_i x u_i y v_i \, .$$

This is derived from a more general construction of dendriform algebras from associative algebras equipped with a Baxter operator, given in Proposition 4.5. (Baxter operators should not be confused with Yang-Baxter operators, see Remark 4.4.)

As a main application of this construction, we work out the dendriform algebra structure associated to the Drinfeld double of an ϵ-bialgebra A. This construction, introduced in [1] and reviewed here in Section 2, produces a quasitriangular ϵ-bialgebra structure on the space $(A \otimes A^*) \oplus A \oplus A^*$. We provide explicit formulas for the resulting dendriform structure in Theorem 4.9. This is one of the main results of this paper. It turns out that the subspace $A \otimes A^*$ is closed under the dendriform operations. The resulting dendriform algebra structure on the space $\mathsf{End}(A)$ of linear endomorphisms of A is (Corollary 4.14)

$$T \succ S = (id * T * id)S + (id * T)(S * id) \quad \text{and} \quad T \prec S = T(id * S * id) + (T * id)(id * S)\, .$$

In this formula, T and S are arbitrary endomorphisms of A, $T * S = \mu(T \otimes S)\Delta$ is the convolution, and the concatenation of endomorphisms denotes composition. When A is a quasitriangular ϵ-bialgebra, our results give dendriform structures on A and $\mathsf{End}(A)$. In Proposition 4.13, we show that they are related by a canonical morphism of dendriform algebras $\mathsf{End}(A) \to A$.

Other properties of the construction of dendriform algebras are given in Section 4. In particular, it is shown that the constructions of pre-Lie algebras from ϵ-bialgebras and of dendriform algebras from quasitriangular ϵ-bialgebras are compatible, in the sense that the diagram

$$\begin{array}{ccc} \text{Quasitriangular } \epsilon\text{-bialgebras} & \longrightarrow & \epsilon\text{-bialgebras} \\ \big\downarrow & & \big\downarrow \\ \text{Dendriform algebras} & \longrightarrow & \text{Pre-Lie algebras} \end{array}$$

commutes.

This paper also introduces the appropriate notion of modules over infinitesimal bialgebras. These are called infinitesimal Hopf modules, abbreviated ϵ-Hopf bimodules. They are defined in Section 2. In the same section, it is shown that ϵ-Hopf bimodules are precisely modules over the double, when the ϵ-bialgebra is finite dimensional (Theorem 2.5), and that any module can be turned into an ϵ-Hopf bimodule, when the ϵ-bialgebra is quasitriangular (Proposition 2.7).

The constructions of dendriform and pre-Lie algebras are extended to the corresponding categories of bimodules in Section 5. A commutative diagram of the form

$$\begin{array}{ccc} \text{Associative bimodules} & \longrightarrow & \epsilon\text{-Hopf bimodules} \\ \big\downarrow & & \big\downarrow \\ \text{Dendriform bimodules} & \longrightarrow & \text{Pre-Lie bimodules} \end{array}$$

is obtained.

A brace algebra is a space B equipped with a family of higher degree operations satisfying certain axioms(6.1). Brace algebras originated in work of Kadeishvili [18], Getzler [16] and Gerstenhaber and Voronov [14, 15]. In this paper we deal with the ungraded, unsigned version of these objects, as in the recent works of Chapoton [6] and Ronco [26]. Brace algebras sit between dendriform and pre-Lie; as explained in [6, 26], the functor from dendriform to pre-Lie algebras factors through the category of brace algebras. Following a suggestion of Ronco, we show in Section 6 that the construction of pre-Lie algebras from ϵ-bialgebras can be refined accordingly. We associate a brace algebra to any ϵ-bialgebra (Theorem 6.2) and obtain a commutative diagram

$$\begin{array}{ccccc} \text{Quasitriangular } \epsilon\text{-bialgebras} & \longrightarrow & \epsilon\text{-bialgebras} & & \\ \downarrow & & \downarrow & \searrow & \\ \text{Dendriform algebras} & \longrightarrow & \text{Brace algebras} & \longrightarrow & \text{Pre-Lie algebras} \end{array}$$

The brace algebra associated to the ϵ-bialgebra of divided differences is explicitly described in Example 6.3. The higher braces are given by

$$\langle \mathbf{x}^{p_1}, \ldots, \mathbf{x}^{p_n}; \mathbf{x}^r \rangle = \binom{r}{n} \mathbf{x}^{r+p_1+\cdots+p_n-n} ,$$

where $\binom{r}{n}$ is the binomial coefficient.

In Appendix A we construct a certain monoidal category of algebras for which the comonoid objects are precisely ϵ-bialgebras, and we discuss how ϵ-bialgebras differ from bimonoid objects in certain related braided monoidal categories.

In Appendix B we study certain special features of counital ϵ-bialgebras. We construct another monoidal category of algebras and show that comonoid objects in this category are precisely counital ϵ-bialgebras (Proposition B.5). The relation to the constructions of Appendix A is explained. We also describe counital ϵ-Hopf modules in terms of this monoidal structure (Proposition B.9).

Notation and basic terminology. All spaces and algebras are over a fixed field k, often omitted from the notation. Sum symbols are omitted from Sweedler's notation: we write $\Delta(a) = a_1 \otimes a_2$ when Δ is a coassociative comultiplication, and similarly for comodule structures. The composition of maps $f : U \to V$ with $g : V \to W$ is denoted by $gf : U \to W$.

2. Infinitesimal modules over infinitesimal bialgebras

An infinitesimal bialgebra (abbreviated ϵ-bialgebra) is a triple (A, μ, Δ) where (A, μ) is an algebra, (A, Δ) is a coalgebra, and for each $a, b \in A$,

$$\Delta(ab) = ab_1 \otimes b_2 + a_1 \otimes a_2 b \,. \tag{2.1}$$

We do not require the algebra to be unital or the coalgebra to be counital.

A derivation of an algebra A with values in a A-bimodule M is a linear map $D : A \to M$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \;\; \forall \; a, b \in A \;.$$

We view $A \otimes A$ as an A-bimodule via

$$a \cdot (b \otimes c) = ab \otimes c \text{ and } (b \otimes c) \cdot a = b \otimes ca \,.$$

A coderivation from a C-bicomodule M to a coalgebra C is a map $D : M \to C$ such that

$$\Delta D = (id_C \otimes D)t + (D \otimes id_C)s \;,$$

where $t : M \to C \otimes M$ and $s : M \to M \otimes C$ are the bicomodule structure maps [Doi]. We view $C \otimes C$ as a C-bicomodule via

$$t = \Delta \otimes id_C \text{ and } s = id_C \otimes \Delta \,.$$

The compatibility condition (2.1) may be written as

$$\Delta\mu = (\mu \otimes id_A)(id_A \otimes \Delta) + (id_A \otimes \mu)(\Delta \otimes id_A)$$

This says that $\Delta : A \to A \otimes A$ is a derivation of the algebra (A, μ) with values in the A-bimodule $A \otimes A$, or equivalently, that $\mu : A \otimes A \to A$ is a coderivation from the A-bicomodule $A \otimes A$ with values in the coalgebra (A, Δ).

Definition 2.1. Let (A, μ, Δ) be an ϵ-bialgebra. A left infinitesimal Hopf module (abbreviated ϵ-Hopf module) over A is a space M endowed with a left A-module structure $\lambda : A \otimes M \to M$ and a left A-comodule structure $\Lambda : M \to A \otimes M$, such that

$$\Lambda\lambda = (\mu \otimes id_M)(id_A \otimes \Lambda) + (id_A \otimes \lambda)(\Delta \otimes id_M)\,.$$

We will often write

$$\lambda(a \otimes m) = am \text{ and } \Lambda(m) = m_{-1} \otimes m_0$$

The compatibility condition above may be written as $\Lambda(am) = a\Lambda(m) + \Delta(a)m$, or more explicitly,

$$(am)_{-1} \otimes (am)_0 = am_{-1} \otimes m_0 + a_1 \otimes a_2 m\,, \quad \text{for each } a \in A \text{ and } m \in M. \tag{2.2}$$

The notion of ϵ-Hopf modules bears a certain analogy to the notion of Hopf modules over ordinary Hopf algebras. The basic examples of Hopf modules from [25, 1.9.2-3] admit the following versions in the context of ϵ-bialgebras.

Examples 2.2. Let (A, μ, Δ) be an ϵ-bialgebra.

(1) A itself is an ϵ-Hopf module via μ and Δ, precisely by definition of ϵ-bialgebra.
(2) More generally, for any space V, $A \otimes V$ is an ϵ-Hopf module via

$$\mu \otimes id : A \otimes A \otimes V \to A \otimes V \text{ and } \Delta \otimes id : A \otimes V \to A \otimes A \otimes V\,.$$

(3) A more interesting example follows. Assume that the coalgebra (A, Δ) admits a counit $\eta : A \to k$. Let N be a left A-module. Then there is an ϵ-Hopf module structure on the space $A \otimes N$ defined by

$$a \cdot (a' \otimes n) = aa' \otimes n + \eta(a')\, a_1 \otimes a_2 n \quad \text{and} \quad \Lambda(a \otimes n) = a_1 \otimes a_2 \otimes n\,.$$

This can be checked by direct calculations. A more conceptual proof will be given later (Corollary B.10). Note that if N is a trivial A-module ($an \equiv 0$) then this structure reduces to that of example 2.

When H is a finite dimensional (ordinary) Hopf algebra, left Hopf modules over H are precisely left modules over the *Heisenberg double* of H [25, Examples 4.1.10 and 8.5.2].

There is an analogous result for infinitesimal bialgebras which, as it turns out, involves the Drinfeld double of ϵ-bialgebras.

We first recall the construction of the Drinfeld double $D(A)$ of a finite dimensional ϵ-bialgebra (A, μ, Δ) from [1, Section 7]. Consider the following version of the dual of A

$$A' := (A^*, \Delta^{*^{op}}, -\mu^{*^{cop}})\,.$$

Explicitly, the structure on A' is:

$$(f \cdot g)(a) = g(a_1)f(a_2) \;\forall\; a \in A,\; f, g \in A' \text{ and} \tag{2.3}$$

$$\Delta(f) = f_1 \otimes f_2 \iff f(ab) = -f_2(a)f_1(b) \;\forall\; f \in A',\; a, b \in A\,. \tag{2.4}$$

Below we always refer to this structure when dealing with multiplications or comultiplications of elements of A'. Consider also the actions of A' on A and A on A' defined by

$$f \rightarrow a = f(a_1)a_2 \text{ and } f \leftarrow a = -f_2(a)f_1 \tag{2.5}$$

or equivalently

$$g(f \rightarrow a) = (gf)(a) \text{ and } (f \leftarrow a)(b) = f(ab)\,. \tag{2.6}$$

Proposition 2.3. *Let A be a finite dimensional ϵ-bialgebra, consider the vector space*

$$D(A) := (A \otimes A') \oplus A \oplus A'$$

and denote the element $a \otimes f \in A \otimes A' \subseteq D(A)$ by $a \bowtie f$. Then $D(A)$ admits a unique ϵ-bialgebra structure such that:

(a) *A and A' are subalgebras, $a \cdot f = a \bowtie f$, $f \cdot a = f \rightarrow a + f \leftarrow a$, and*
(b) *A and A' are subcoalgebras.*

Proof. See [3, Theorem 7.3]. □

We will make use of the following universal property of the double.

Proposition 2.4. *Let A be a finite dimensional ϵ-bialgebra, B an algebra and $\rho : A \to B$ and $\rho' : A' \to B$ morphisms of algebras such that $\forall\ a \in A,\ f \in A'$,*

$$\rho'(f)\rho(a) = \rho(f \rightarrow a) + \rho'(f \leftarrow a)\ . \tag{2.7}$$

Then there exists a unique morphism of algebras $\hat{\rho} : D(A) \to B$ such that $\hat{\rho}_{|A} = \rho$ and $\hat{\rho}_{|'_A} = \rho'$.

Proof. This follows from Propositions 6.5 and 7.1 in [1]. □

We can now show that ϵ-Hopf modules are precisely modules over the double.

Theorem 2.5. *Let A be a finite dimensional ϵ-bialgebra and M a space. If M is a left ϵ-Hopf module over A via $\lambda(a \otimes m) = am$ and $\Lambda(m) = m_{-1} \otimes m_0$, then M is a left module over $D(A)$ via*

$$a \cdot m = am, \quad f \cdot m = f(m_{-1})m_0 \quad and \quad (a \bowtie f) \cdot m = f(m_{-1})am_0\,.$$

Conversely, if M is a left module over $D(A)$, then M is a left ϵ-Hopf module over A and the structures are related as above.

Proof. Suppose first that M is a left ϵ-Hopf module over A.

Since (M, Λ) is a left A-comodule, it is also a left A'-module via $f \cdot m := f(m_{-1})m_0$. Let $\rho : A \to \mathsf{End}(M)$ and $\rho' : A' \to \mathsf{End}(M)$ be the morphisms of algebras corresponding to the left module structures:

$$\rho(a)(m) = am, \quad \rho'(f)(m) = f(m_{-1})m_0\,.$$

We will apply Proposition 2.4 to deduce the existence of a morphism of algebras $\hat{\rho} : D(A) \to \mathsf{End}(M)$ extending ρ and ρ'. We need to check (2.7). We have

$$\begin{aligned} \rho'(f)\rho(a)(m) = f\big((am)_{-1}\big)(am)_0 &\overset{(2.2)}{=} f(a_1)a_2m + f(am_{-1})m_0 \\ &\overset{(2.5,\,2.6)}{=} (f \rightarrow a)m + (f \leftarrow a)(m_{-1})m_0 \\ &= \rho(f \rightarrow a)(m) + \rho'(f \leftarrow a)(m) \end{aligned}$$

as needed. Thus, $\hat{\rho}$ exists and M becomes a left $D(A)$-module via $\alpha \cdot m = \hat{\rho}(\alpha)(m)$. Since $\hat{\rho}$ extends ρ and ρ', we have

$$a \cdot m = \rho(a)(m) = am, \quad f \cdot m = \rho'(f)(m) = f(m_{-1})m_0$$

and, from the description of the multiplication in $D(A)$ in Proposition 2.3,

$$(a \bowtie f) \cdot m = \hat{\rho}(af)(m) = \rho(a)\rho'(f)(m) = f(m_{-1})am_0\,.$$

This completes the proof of the first assertion.

Conversely, if M is a left $D(A)$-module, then restricting via the morphisms of algebras $A \hookrightarrow D(A)$ and $A' \hookrightarrow D(A)$, M becomes a left A-module and left A'-module. As above, the latter structure is equivalent to a left A-comodule structure on M. From the associativity axiom

$$f \cdot (a \cdot m) = (fa) \cdot m = (f \rightarrow a) \cdot m + (f \leftarrow a) \cdot m$$

we deduce

$$f\big((am)_{-1}\big)(am)_0 = f(a_1)a_2m + f(am_{-1})m_0\,.$$

Since this holds for every $f \in A'$, we obtain the ϵ-Hopf module Axiom (2.2). Also,

$$(a \bowtie f) \cdot m = (af) \cdot m = a \cdot (f \cdot m) = f(m_{-1})am_0\,,$$

so the structures of left module over $D(A)$ and left ϵ-Hopf module over A are related as stated. □

We close the section by showing that when A is a quasitriangular ϵ-bialgebra, any A-module carries a natural structure of ϵ-Hopf module over A.

We first recall the definition of quasitriangular ϵ-bialgebras. Let A be an associative algebra. An element $r = \sum_i u_i \otimes v_i \in A \otimes A$ is a solution of the associative Yang-Baxter equation [1, Section 5] if

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0 \tag{2.8}$$

or, more explicitly,

$$\sum_{i,j} u_iu_j \otimes v_j \otimes v_i - \sum_{i,j} u_i \otimes v_iu_j \otimes v_j + \sum_{i,j} u_j \otimes u_i \otimes v_iv_j = 0\,.$$

This condition implies that the *principal* derivation $\Delta : A \to A \otimes A$ defined by

$$\Delta(a) = r \cdot a - a \cdot r = \sum_i u_i \otimes v_ia - \sum_i au_i \otimes v_i\,, \tag{2.9}$$

is coassociative [1, Proposition 5.1]. Thus, endowed with this comultiplication, A becomes an ϵ-bialgebra. We refer to the pair (A, r) as a quasitriangular ϵ-bialgebra [1, Definition 5.3].

Remark 2.6. In our previous work [1, 3], we have used the comultiplication

$$-\Delta(a) = a \cdot r - r \cdot a = \sum_i au_i \otimes v_i - \sum_i u_i \otimes v_ia$$

instead of Δ. Both Δ and $-\Delta$ endow A with a structure of ϵ-bialgebra, and there is no essential difference in working with one or the other. The choice we adopt in (2.9), however, is more convenient for the purposes of this work, particularly in relating quasitriangular ϵ-bialgebras and their bimodules to dendriform algebras and their bimodules (Sections 4 and 5).

It is then necessary to make the corresponding sign adjustments to the results on quasitriangular ϵ-bialgebras from [1, 3] before applying them in the present context. For instance, Proposition 5.5 in [1] translates as

$$\sum_{i,j} u_i \otimes u_j \otimes v_jv_i = r_{23}r_{13} = (\Delta \otimes id)(r) = \Delta(u_i) \otimes v_i\,. \tag{2.10}$$

Proposition 2.7. *Let (A, r) be a quasitriangular ϵ-bialgebra and M a left A-module. Then M becomes a left ϵ-Hopf module over A via $\Lambda : M \to A \otimes M$,*

$$\Lambda(m) = \sum_i u_i \otimes v_im\,.$$

Proof. We first check that Λ is coassociative, i.e., $(id\otimes\Lambda)\Lambda = (\Delta\otimes id)\Lambda$. We have

$$(id\otimes\Lambda)\Lambda(m) = \sum_i u_i\otimes\Lambda(v_i m) = \sum_{i,j} u_i\otimes u_j\otimes v_j v_i m$$

and

$$(\Delta\otimes id)\Lambda(m) = \sum \Delta(u_i)\otimes v_i m\,.$$

According to (2.10), these two expressions agree.

It only remains to check Axiom (2.2). Since $\Delta(a) = \sum_i u_i\otimes v_i a - \sum_i a u_i\otimes v_i$, we have

$$\Delta(a)m + a\Lambda(m) = \sum_i u_i\otimes v_i am - \sum_i au_i\otimes v_i m + \sum_i au_i\otimes v_i m = \sum_i u_i\otimes v_i am = \Lambda(am)\,,$$

as needed. □

Remark 2.8. If A is a finite dimensional quasitriangular ϵ-bialgebra, then there is a canonical morphism of ϵ-bialgebras $\pi : D(A) \to A$, which is the identity on A [1, Proposition 7.5]. Therefore, any left A-module M can be first made into a left $D(A)$-module by restriction via π, and then, by Theorem 2.5, into a left ϵ-Hopf module over A. It is easily seen that this structure coincides with the one of Proposition 2.7. Note that the construction of the latter proposition is more general, since it does not require finite dimensionality of A.

3. Pre-Lie algebras

Definition 3.1. A (left) pre-Lie algebra is a vector space P together with a map $\circ : P\otimes P \to P$ such that

$$x\circ(y\circ z) - (x\circ y)\circ z = y\circ(x\circ z) - (y\circ x)\circ z\,. \tag{3.1}$$

There is a similar notion of right pre-Lie algebras. In this paper, we will only deal with left pre-Lie algebras and we will refer to them simply as pre-Lie algebras.

Defining a new operation $P\otimes P \to P$ by $[x,y] = x\circ y - y\circ x$ one obtains a Lie algebra structure on P [13, Theorem 1].

Next we show that every ϵ-bialgebra A gives rise to a structure of pre-Lie algebra, and hence also of Lie algebra, on the underlying space of A.

Theorem 3.2. *Let (A,μ,Δ) be an ϵ-bialgebra. Define a new operation on A by*

$$a\circ b = b_1 a b_2\,. \tag{3.2}$$

Then $(A,\circ)$ is a pre-Lie algebra.

Proof. By repeated use of (2.1) we find

$$\Delta(abc) = ab\cdot\Delta(c) + \Delta(ab)\cdot c = abc_1\otimes c_2 + ab_1\otimes b_2c + a_1\otimes a_2bc\,.$$

Together with coassociativity this gives

$$\Delta(c_1bc_2) = c_1bc_2\otimes c_3 + c_1b_1\otimes b_2c_2 + c_1\otimes c_2bc_3\,.$$

Combining this with (3.2) we obtain

$$a\circ(b\circ c) = a\circ(c_1bc_2) = c_1bc_2ac_3 + c_1b_1ab_2c_2 + c_1ac_2bc_3\,.$$

On the other hand,

$$(a\circ b)\circ c = (b_1ab_2)\circ c = c_1b_1ab_2c_2\,.$$

Therefore,

$$a\circ(b\circ c) - (a\circ b)\circ c = c_1bc_2ac_3 + c_1ac_2bc_3\,.$$

Since this expression is invariant under $a \leftrightarrow b$, Axiom (3.1) holds and $(A, \circ)$ is a pre-Lie algebra. □

For a vector space V, let $\mathsf{gl}(V)$ denote the space of all linear maps $V \to V$, viewed as a Lie algebra under the commutator bracket $[T, S] = TS - ST$.

If P is a pre-Lie algebra and $x \in P$, let $L_x : P \to P$ be $L_x(y) = x \circ y$. The map $L : P \to \mathsf{gl}(P)$, $x \mapsto L_x$ is a morphism of Lie algebras. This statement is just a reformulation of Axiom (3.1).

In the case when the pre-Lie algebra comes from an ϵ-bialgebra (A, μ, Δ), more can be said about this canonical map. Let $\mathsf{Der}(A, \mu)$ denote the space of all derivations $D : A \to A$ of the associative algebra A. Recall that this is a Lie subalgebra of $\mathsf{gl}(A)$.

Proposition 3.3. *Let (A, μ, Δ) be an ϵ-bialgebra and consider the associated pre-Lie and Lie algebra structures on A. The canonical morphism of Lie algebras $L : (A, [\ ,\ ]) \to \mathsf{gl}(A)$ actually maps to the Lie subalgebra $\mathsf{Der}(A, \mu)$ of $\mathsf{gl}(A)$.*

Proof. We must show that each $L_c \in \mathsf{gl}(A)$ is a derivation of the associative algebra A. We have $\Delta(ab) \overset{(2.1)}{=} ab_1 \otimes b_2 + a_1 \otimes a_2 b$ and hence

$$L_c(ab) = c \circ (ab) \overset{(3.2)}{=} ab_1 c b_2 + a_1 c a_2 b \overset{(3.2)}{=} a(c \circ b) + (c \circ a)b = aL_c(b) + L_c(a)b \ ,$$

as needed.

□

Examples 3.4.

(1) Consider the ϵ-bialgebra of *divided differences*. This is the algebra $k[\mathbf{x}, \mathbf{x}^{-1}]$ of Laurent polynomials, with $\Delta(f(\mathbf{x})) = \frac{f(\mathbf{x}) - f(\mathbf{y})}{\mathbf{x} - \mathbf{y}}$. This was the example that motivated Joni and Rota to abstract the notion of ϵ-bialgebras [17, Section XII]. More explicitly,

$$\Delta(\mathbf{x}^n) = \sum_{i=0}^{n-1} \mathbf{x}^i \otimes \mathbf{x}^{n-1-i} \quad \text{and} \quad \Delta(\frac{1}{\mathbf{x}^n}) = -\sum_{i=1}^{n} \frac{1}{\mathbf{x}^i} \otimes \frac{1}{\mathbf{x}^{n+1-i}} \ , \quad \text{for } n \geq 0.$$

The corresponding pre-Lie algebra structure is

$$\mathbf{x}^m \circ \mathbf{x}^n = n\mathbf{x}^{m+n-1} \ , \quad \text{for any } n \in \mathbb{Z},$$

and the Lie algebra structure on $k[\mathbf{x}, \mathbf{x}^{-1}]$ is

$$[\mathbf{x}^m, \mathbf{x}^n] = (n - m)\mathbf{x}^{m+n-1} \quad \text{for } n,\ m \in \mathbb{Z}.$$

This is the so called Witt Lie algebra. The canonical map $k[\mathbf{x}, \mathbf{x}^{-1}] \to \mathsf{Der}(k[\mathbf{x}, \mathbf{x}^{-1}], \mu)$ of Proposition 3.3 sends $\mathbf{x}^m$ to $\mathbf{x}^m \frac{d}{d\mathbf{x}}$, so it is an isomorphism of Lie algebras.

(2) The algebra of matrices $M_2(k)$ is an ϵ-bialgebra under

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

[1, Example 2.3.7]. One finds easily that the corresponding Lie algebra splits as a direct sum of Lie algebras

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{o}$$

where $\mathfrak{h} = k\{x, y, z\}$ is the 3-dimensional *Heisenberg* algebra

$$\{x, y\} = z, \quad \{x, z\} = \{y, z\} = 0 \ ,$$

and $\mathfrak{o} = k\{i\}$ is the 1-dimensional Lie algebra. To realize this isomorphism explicitly, one may take

$$x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ .$$

(3) The *path algebra* of a quiver carries a canonical ϵ-bialgebra structure [1, Example 2.3.2]. Let Q be an arbitrary quiver (i.e., an oriented graph). Let Q_n be the set of paths α in Q of length n:

$$\alpha : e_0 \xrightarrow{a_1} e_1 \xrightarrow{a_2} e_2 \xrightarrow{a_3} \dots e_{n-1} \xrightarrow{a_n} e_n \ .$$

In particular, Q_0 is the set of vertices and Q_1 is the set of arrows. Recall that the path algebra of Q is the space $kQ = \oplus_{n=0}^{\infty} kQ_n$ where multiplication is concatenation of paths whenever possible; otherwise is zero. The comultiplication is defined on a path $\alpha = a_1 a_2 \dots a_n$ as above by

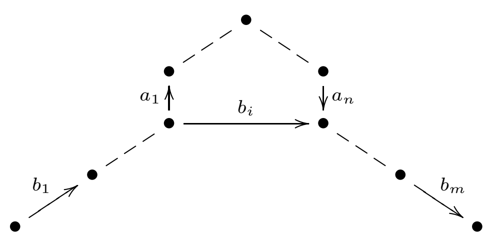
$$\Delta(\alpha) = e_0 \otimes a_2 a_3 \dots a_n + a_1 \otimes a_3 \dots a_n + \dots + a_1 \dots a_{n-1} \otimes e_n \ .$$

In particular, $\Delta(e) = 0$ for every vertex e and $\Delta(a) = e_0 \otimes e_1$ for every arrow $e_0 \xrightarrow{a} e_1$.

In order to describe the corresponding pre-Lie algebra structure on kQ, consider pairs (α, b) where α is a path from e_0 to e_n (as above) and b is an arrow from e_0 to e_n. Let us call such a pair a *shortcut*. The pre-Lie algebra structure on kQ is

$$\alpha \circ \beta = \sum_{b_i \in \beta} b_1 \dots b_{i-1} \alpha b_{i+1} \dots b_m \,,$$

where the sum is over all arrows b_i in the path $\beta = b_1 \dots b_m$ such that (α, b_i) is a shortcut.



A *biderivation* of an ϵ-bialgebra (A, μ, Δ) is a map $B : A \to A$ that is both a derivation of (A, μ) and a coderivation of (A, Δ), i.e.,

$$B(ab) = aB(b) + B(a)b \ \text{ and } \ \Delta(B(a)) = a_1 \otimes B(a_2) + B(a_1) \otimes a_2 \,. \tag{3.3}$$

A derivation of a pre-Lie algebra is a map $D : P \to P$ such that

$$D(x \circ y) = x \circ D(y) + D(x) \circ y \,.$$

Such a map D is always a derivation of the associated Lie algebra.

Proposition 3.5. *Let B be a biderivation of an ϵ-bialgebra A. Then B is a derivation of the associated pre-Lie algebra (and hence also of the associated Lie algebra).*

Proof. We have

$$B(a) \circ b = b_1 B(a) b_2 \ \text{ and } \ a \circ B(b) = b_1 a B(b_2) + B(b_1) a b_2 \,.$$

Hence,

$$B(a) \circ b + a \circ B(b) = b_1 B(a) b_2 + b_1 a B(b_2) + B(b_1) a b_2 = B(b_1 a b_2) = B(a \circ b) \,.$$

□

The construction of a pre-Lie algebra from and ϵ-bialgebra can be extended to the categories of modules. This will be discussed in the appropriate generality in Section 5. A first result in this direction is discussed next.

Let $(P, \circ)$ be a pre-Lie algebra. A left P-module is a space M together with a map $P \otimes M \to M$, $x \otimes m \mapsto x \circ m$, such that

$$x \circ (y \circ m) - (x \circ y) \circ m = y \circ (x \circ m) - (y \circ x) \circ m \,. \tag{3.4}$$

Proposition 3.6. *Let A be an ϵ-bialgebra and M a left ϵ-Hopf module over A via*

$$\lambda(a \otimes m) = am \quad \text{and} \quad \Lambda(m) = m_{-1} \otimes m_0 \,.$$

Then M is a left pre-Lie module over the pre-Lie algebra $(A, \circ)$ of Theorem 3.2 via

$$a \circ m = m_{-1} a m_0 \,.$$

Proof. We first compute

$$\begin{aligned}\Lambda(a \circ m) = \Lambda(m_{-1}am) &\overset{(2.2)}{=} \Delta(m_{-1}a)m_0 + m_{-1}a\Lambda(m_0) \\ &\overset{(2.1)}{=} m_{-1} \otimes m_{-1}am_0 + m_{-1}a_1 \otimes a_2 m_0 + m_{-2}am_{-1} \otimes m_0 \,,\end{aligned}$$

where we have used the coassociativity axiom for the comodule structure Λ. It follows that

$$b \circ (a \circ m) = m_{-2}bm_{-1}am_0 + m_{-1}a_1ba_2m_0 + m_{-2}am_{-1}bm_0 \,.$$

On the other hand,

$$(b \circ a) \circ m = m_{-1}(b \circ a)m_0 \overset{(3.2)}{=} m_{-1}a_1ba_2m_0 \,.$$

Therefore,

$$b \circ (a \circ m) - (b \circ a) \circ m = m_{-2}bm_{-1}am_0 + m_{-2}am_{-1}bm_0 \,.$$

Since this expression is invariant under $a \leftrightarrow b$, Axiom (3.4) holds. □

Remark 3.7. Since the notion of ϵ-bialgebras is self-dual [1, Section 2], one should expect dual constructions to those of Theorem 3.2 and Propositions 3.5 and 3.6. This is indeed the case. Namely, if A is an arbitrary ϵ-bialgebra, then the map $\gamma : A \to A \otimes A$ defined by

$$\gamma(a) = a_2 \otimes a_1 a_3$$

endows A with a structure of *left pre-Lie coalgebra*. Also, if $B : A \to A$ is a biderivation of A then it is also a coderivation of (A, γ). Moreover, if M is a left ϵ-Hopf module over A, then M is left pre-Lie comodule over (A, γ) via $\psi : M \to A \otimes M$ defined by

$$\psi(m) = m_{-1} \otimes m_{-2} m_0 \,.$$

If A is an ϵ-bialgebra, then A carries structures of pre-Lie algebra and pre-Lie coalgebra, as just explained. Hence, it also carries structures of Lie algebra and Lie coalgebra, by

$$[a, b] = b_1 a b_2 - a_1 b a_2 \quad \text{and} \quad \delta(a) = a_2 \otimes a_1 a_3 - a_1 a_3 \otimes a_2 \,.$$

In general, these structures are not compatible, in the sense that they do *not* define a structure of Lie bialgebra on A.

4. Dendriform algebras

Definition 4.1. A dendriform algebra is a vector space D together with maps $\succ : D \otimes D \to D$ and $\prec : D \times D \to D$ such that

$$\begin{aligned}&(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z) \\ &x \succ (y \prec z) = (x \succ y) \prec z \\ &x \succ (y \succ z) = (x \prec y) \succ z + (x \succ y) \succ z \,.\end{aligned} \tag{4.1}$$

Dendriform algebras were introduced by Loday [20, Chapter 5]. There is also a notion of *dendriform trialgebras*, which involves three operations [23]. When it is necessary to distinguish between these two notions, one uses the name *dendriform dialgebras* to refer to what in this paper (and in [20]) are called dendriform algebras. Since only dendriform algebras (in the sense of Definition 4.1) will be considered in this paper, this usage will not be adopted.

Let $(D, \succ, \prec)$ be a dendriform algebra. Defining $x \cdot y = x \succ y + x \prec y$ one obtains an associative algebra structure on D. In addition, defining

$$x \circ y = x \succ y - y \prec x \tag{4.2}$$

one obtains a (left) pre-Lie algebra structure on D. Moreover, the Lie algebras canonically associated to $(D, \cdot)$ and $(D, \circ)$ coincide; namely,

$$x \cdot y - y \cdot x = x \succ y + x \prec y - y \succ x - y \prec x = x \circ y - y \circ x\,.$$

If $f : D \to D'$ is a morphism of dendriform algebras, then it is also a morphism with respect to any of the other three structures on D. The situation may be summarized by means of the following commutative diagram of categories

$$\begin{array}{ccc} \text{Dendriform algebras} & \longrightarrow & \text{Pre-Lie algebras} \\ \downarrow & & \downarrow \\ \text{Associative algebras} & \longrightarrow & \text{Lie algebras} \end{array}$$

Recall the notion of quasitriangular ϵ-bialgebras from Section 2. In this section we show that there is a commutative diagram as follows:

$$\begin{array}{ccc} \text{Quasitriangular } \epsilon\text{-bialgebras} & \longrightarrow & \epsilon\text{-bialgebras} \\ \downarrow & & \downarrow \\ \text{Dendriform algebras} & \longrightarrow & \text{Pre-Lie algebras} \end{array} \tag{4.3}$$

In this diagram, the right vertical arrow is the functor constructed in Section 3, the bottom horizontal arrow is the construction just discussed (4.2) and the top horizontal arrow is simply the inclusion. It remains to discuss the construction of a dendriform algebra from a quasitriangular ϵ-bialgebra, and to verify the commutativity of the diagram.

This construction is best understood from the point of view of *Baxter operators*.

Definition 4.2. Let A be an associative algebra. A Baxter operator is a map $\beta : A \to A$ that satisfies the condition

$$\beta(x)\beta(y) = \beta\Big(x\beta(y) + \beta(x)y\Big)\,. \tag{4.4}$$

Baxter operators arose in probability theory [5] and were a subject of interest to Gian-Carlo Rota [27, 28].

We start by recalling a basic result from [2], which provides us with the examples of Baxter operators that are most relevant for our present purposes.

Proposition 4.3. *Let $r = \sum_i u_i \otimes v_i$ be a solution of the associative Yang-Baxter equation* (2.8) *in an associative algebra A. Then the map $\beta : A \to A$ defined by*

$$\beta(x) = \sum_i u_i x v_i$$

is a Baxter operator.

Proof. Replacing the tensor symbols in the associative Yang-Baxter equation (2.8) by x and y one obtains precisely (4.4). □

Remark 4.4. The associative Yang-Baxter equation is analogous to the *classical Yang-Baxter equation*, which is named after C. N. Yang and R. J. Baxter. Baxter operators, on the other hand, are named after Glen Baxter.

The following result provides the second step in the construction of dendriform algebras from quasitriangular ϵ-bialgebras.

Proposition 4.5. *Let A be an associative algebra and $\beta : A \to A$ a Baxter operator. Define new operations on A by*

$$x \succ y = \beta(x)y \quad \textit{and} \quad x \prec y = x\beta(y)\ .$$

Then $(A, \succ, \prec)$ is a dendriform algebra.

Proof. We verify the last axiom in (4.1); the others are similar. We have

$$\begin{aligned} x \succ (y \succ z) &= \beta(x)(y \succ z) = \beta(x)\beta(y)z \\ &\overset{(4.4)}{=} \beta\Big(x\beta(y) + \beta(x)y\Big)z = \beta(x \prec y + x \succ y)z \\ &= \beta(x \prec y)z + \beta(x \succ y)z = (x \prec y) \succ z + (x \succ y) \succ z\,. \end{aligned}$$

□

Extensions of the above result appear in [2, Propositions 5.1 and 5.2].

Finally, we have the desired construction of dendriform algebras from quasitriangular ϵ-bialgebras.

Theorem 4.6. *Let (A, r) be a quasitriangular ϵ-bialgebra, $r = \sum_i u_i \otimes v_i$. Define new operations on A by*

$$x \succ y = \sum_i u_i x v_i y \quad \textit{and} \quad x \prec y = \sum_i x u_i y v_i\,. \tag{4.5}$$

Then, $(A, \succ, \prec)$ is a dendriform algebra.

Proof. Combine Propositions 4.3 and 4.5. □

A morphism between quasitriangular ϵ-bialgebras (A, r) and (A', r') is a morphism of algebras $f : A \to A'$ such that $(f \otimes f)(r) = r'$. Clearly, such a map f preserves the dendriform structures on A and A'. Thus, we have constructed a functor from quasitriangular ϵ-bialgebras to dendriform algebras.

We briefly discuss the functoriality of the construction with respect to derivations.

A derivation of a quasitriangular ϵ-bialgebra (A, r) is a map $D : A \to A$ that is a derivation of the associative algebra A such that

$$(D \otimes id + id \otimes D)(r) = 0\,.$$

This implies that D is a biderivation of the ϵ-bialgebra associated to (A, r), in the sense of (3.3). In fact, it is easy to see that D is a biderivation if and only $(D \otimes id + id \otimes D)(r)$ is an invariant element in the A-bimodule $A \otimes A$. These two conditions are analogous to the ones encountered in the definition of quasitriangular and coboundary ϵ-bialgebras [3, Section 1]. The stronger condition guarantees the following:

Proposition 4.7. *Let D be a derivation of a quasitriangular ϵ-bialgebra (A, r). Then D is also a derivation of the associated dendriform algebra, i.e.,*

$$D(a \succ b) = a \succ D(b) + D(a) \succ b \quad \textit{and} \quad D(a \prec b) = a \prec D(b) + D(a) \prec b\,.$$

Proof. Similar to the other proofs in this section. □

It remains to verify the commutativity of diagram (4.3). Starting from (A, r) and going clockwise, we pass through the ϵ-bialgebra with comultiplication $\Delta(b) = \sum_i u_i \otimes v_i b - \sum_i b u_i \otimes v_i$, according to (2.9) (see also Remark 2.6). The associated pre-Lie algebra structure is, by (3.2), $a \circ b = \sum_i u_i a v_i b - \sum_i b u_i a v_i$. According to (4.5), this expression is equal to $a \succ b - b \prec a$, which by (4.2) is the pre-Lie algebra structure obtained by going counterclockwise around the diagram.

Examples 4.8.

(1) Let A be an associative unital algebra and $b \in A$ an element such that $b^2 = 0$. Then, $r := 1 \otimes b$ is a solution of the associative Yang-Baxter equation (2.8) [1, Example 5.4.1]. The corresponding dendriform structure on A is simply

$$x \succ y = xyb \quad \text{and} \quad x \prec y = xby\,.$$

This structure is well defined even if A does not have a unit.

(2) The element $r := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a solution of (2.8) in the algebra of matrices $M_2(k)$ [1, Example 5.4.5.e] and [3, Examples 2.3.1 and 2.8]. The corresponding dendriform structure on $M_2(k)$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \succ \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} az - cx & aw - cy \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \prec \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -az & ax \\ -cz & cx \end{bmatrix}.$$

(3) The most important example is provided by Drinfeld's double, which is a quasitriangular ϵ-bialgebra canonically associated to an arbitrary finite dimensional ϵ-bialgebra. This will occupy the rest of the section.

Let A be a finite dimensional ϵ-bialgebra. Recall the definition of the double $D(A)$ from Section 2. Let $\{e_i\}$ be a linear basis of A and $\{f_i\}$ the dual basis of A^*. Let $r \in D(A) \otimes D(A)$ be the element

$$r = \sum_i e_i \otimes f_i \in A \otimes A^* \subseteq D(A) \otimes D(A)\,.$$

According to [1, Theorem 7.3], $(D(A), r)$ is a quasitriangular ϵ-bialgebra (see Remark 2.6). By Theorem 4.6, there is a dendriform algebra structure on the space $D(A) = (A \otimes A^*) \oplus A \oplus A^*$.

In order to make this structure explicit, we introduce some notation. We identify $A \otimes A^*$ with $\mathsf{End}(A)$ via

$$(a \bowtie f)(b) = f(b)a\,.$$

For each $a \in A$ and $f \in A^*$, define linear endomorphisms of A by

$$\begin{aligned} L_a(x) &= ax & L_f(x) &= f(x_2)x_1 \\ R_a(x) &= xa & R_f(x) &= f(x_1)x_2 \\ P_a(x) &= x_1 a x_2 & P_f(x) &= f(x_2)x_1 x_3\,. \end{aligned}$$

The composition of linear maps $\phi : U \to V$ and $\psi : V \to W$ is denoted by $\psi\phi$, or $\psi(\phi)$, if the expression for ϕ is complicated. This should not be confused with the evaluation of an endomorphism T on an element $a \in A$, denoted by $T(a)$. The convolution of linear endomorphisms T and S of A is

$$T * S = \mu(T \otimes S)\Delta\,.$$

Theorem 4.9. *Let A be an arbitrary ϵ-bialgebra. There is a dendriform structure on the space $\mathsf{End}(A)\oplus A\oplus A^*$, given explicitly as follows. For a, $b\in A$, f, $g\in A^*$ and T, $S\in \mathsf{End}(A)$,*

$$\begin{aligned}
a\succ b &= P_a(b)+R_aL_b & a\prec b &= L_aR_b\\
f\succ a &= P_f(a)+L_fL_a & a\prec f &= L_aL_f\\
f\prec a &= fP_a+R_fR_a & a\succ f &= R_aR_f\\
f\prec g &= fP_g+R_fL_g & f\succ g &= L_fR_g
\end{aligned}$$

$$\begin{aligned}
a\succ T &= P_aT+R_a(T*id) & a\prec T &= L_a(id*T)\\
f\succ T &= P_fT+L_f(T*id) & f\prec T &= f(id*T*id)+R_f(id*T)\\
T\prec a &= TP_a+(T*id)R_a & T\succ a &= (id*T*id)(a)+(id*T)L_a\\
T\prec f &= TP_f+(T*id)L_f & T\succ f &= (id*T)R_f
\end{aligned}$$

$$\begin{aligned}
T\succ S &= (id*T*id)S+(id*T)(S*id)\\
T\prec S &= T(id*S*id)+(T*id)(id*S)
\end{aligned}$$

Proof. Assume that A is finite dimensional, so $D(A)$ and $r=\sum e_i\otimes f_i$ are well defined, and hence there is a dendriform structure on the space $D(A)$. For the details of the infinite dimensional case see Remark 4.10.

We provide the derivations of the first and last formulas, the others are similar. We make use of the ϵ-bialgebra structure of $D(A)$ as described in Proposition 2.3.

For the first formula we have

$$\begin{aligned}
a\succ b=\sum_i e_iaf_ib &= \sum_i e_ia(f_i\rightarrow b)+\sum_i e_ia(f_i\leftarrow b)\\
&\overset{(2.5)}{=}\sum_i e_iaf_i(b_1)b_2+\sum_i e_ia\bowtie(f_i\leftarrow b)\\
&= b_1ab_2+\sum_i e_ia\bowtie(f_i\leftarrow b)\,.
\end{aligned}$$

Now, for any $x\in A$,

$$\sum_i(f_i\leftarrow b)(x)e_ia\overset{(2.6)}{=}\sum_i f_i(bx)e_ia=bxa=R_aL_b(x)\,.$$

Thus, $a\succ b=P_a(b)+R_aL_b$, as claimed.

For the last formula, let $T=a\bowtie f$ and $S=b\bowtie g$. We have

$$\begin{aligned}
T\prec S=(a\bowtie f)\prec(b\bowtie g) &= \sum_i(a\bowtie f)e_i(b\bowtie g)f_i\\
&= \sum_i a(f\rightarrow e_ib)\bowtie gf_i+\sum_i a\bowtie(f\leftarrow e_ib)gf_i\,.
\end{aligned}$$

Hence, for any $x \in A$,

$$\begin{aligned}(T \prec S)(x) &\overset{(2.6)}{=} \sum_i f_i(x_1)g(x_2)a(f \to e_i b) + \sum_i f_i(x_1)g(x_2)(f \leftarrow e_i b)(x_3)a \\ &= g(x_2)a(f \to x_1 b) + g(x_2)(f \leftarrow x_1 b)(x_3)a \\ &\overset{(2.6)}{=} a\big(f \to x_1 S(x_2)\big) + g(x_2)f(x_1 b x_3)a \\ &\overset{(2.5)}{=} f\Big(\big(x_1 S(x_2)\big)_1\Big)a\big(x_1 S(x_2)\big)_2 + f\big(x_1 S(x_2) x_3\big)a \\ &= (T * id)(id * S)(x) + T(id * S * id)(x)\,.\end{aligned}$$

Thus, $T \prec S = (T * id)(id * S) + T(id * S * id)$, as claimed. □

Remark 4.10. The ϵ-bialgebra structure on $D(A)$ and the element $r \in D(A) \otimes D(A)$ are well defined only if A is finite dimensional. However, all formulas in Theorem 4.9 make sense and the theorem holds even if A is infinite dimensional. This may be seen as follows. There is always an algebra structure on the space $\mathsf{End}(A) \oplus A \oplus A^*$, extending that of $D(A)$. Moreover, there is always a Baxter operator on this algebra, well defined by

$$\beta(a) = R_a, \quad \beta(f) = L_f \quad \text{and} \quad \beta(T) = id * T\,.$$

It is easy to see that this coincides with the operator corresponding to r, when A is finite dimensional. In the general case, it may be checked directly that β satisfies (4.4). The result then follows from Proposition 4.5.

Remark 4.11. In order to fully appreciate the symmetry in the previous formulas, the following relations should be kept in mind:

$$\begin{aligned} T * R_a &= R_a(T * id) & T * L_f &= (T * id)L_f \\ L_a * T &= L_a(id * T) & R_f * T &= (id * T)R_f \\ L_a R_b &= R_b L_a & L_f R_g &= R_g L_f \end{aligned}$$

Remark 4.12. Consider the pre-Lie algebra structures on A and A' corresponding to their ϵ-bialgebra structures by means of Theorem 3.2. Since A and A' are ϵ-subbialgebras of $D(A)$ (Proposition 2.3), the functoriality of the construction implies that A and A' are pre-Lie subalgebras of $D(A)$, with respect to the pre-Lie structure associated to the dendriform structure as in (4.2). Let us verify this fact explicitly. The pre-Lie structure on A is

$$a \circ b = a \succ b - b \prec a = P_a(b) + R_a L_b - L_b R_a = P_a(b) = b_1 a b_2\,,$$

as expected. The pre-Lie structure on A' is

$$f \circ g = f \succ g - g \prec f = L_f R_g - g P_f - R_g L_f = -g P_f\,.$$

Thus,

$$(f \circ g)(a) = -g(f(a_2)a_1 a_3) \overset{(2.4)}{=} f(a_2)g_2(a_1)g_1(a_3) \overset{(2.3)}{=} (g_1 f g_2)(a)\,,$$

also as expected (the ϵ-bialgebra structure on A' was described in Section 2).

When A is a quasitriangular ϵ-bialgebra, there are dendriform algebra structures both on A and $\mathsf{End}(A) \oplus A \oplus A^*$, by Theorems 4.6 and 4.9. These two structures are related by a canonical morphism of dendriform algebras.

Proposition 4.13. *Let (A, r) be a quasitriangular ϵ-bialgebra, $r = \sum_i u_i \otimes v_i$. Then, the map*

$$\pi : \mathit{End}(A) \oplus A \oplus A^* \to A, \quad \pi(a) = a, \ \pi(f) = \sum_i f(u_i)v_i \ \text{and} \ \pi(T) = \sum_i T(u_i)v_i$$

is a morphism of dendriform algebras.

Proof. Assume that A is finite dimensional. According to [1, Proposition 7.5], the above formulas define a morphism of ϵ-bialgebras $\pi : D(A) \to A$ (see Remark 2.6). By the functoriality of the construction of dendriform algebras, π is also a morphism of dendriform algebras.

The general case may be obtained by showing that π commutes with the Baxter operators on $\mathsf{End}(A) \oplus A \oplus A^*$ and A. This follows from (2.8) and (2.10). □

The formulas in Theorem 4.9 show that $\mathsf{End}(A)$ is closed under the dendriform operations. Together with Proposition 4.13, this gives the following:

Corollary 4.14. *Let A be an arbitrary ϵ-bialgebra. Then there is a dendriform algebra structure on the space $\mathit{End}(A)$ of linear endomorphisms of A, defined by*

$$T \succ S = (id * T * id)S + (id * T)(S * id) \quad \text{and} \quad T \prec S = T(id * S * id) + (T * id)(id * S)\,.$$

Moreover, if A is quasitriangular, with $r = \sum_i u_i \otimes v_i$, then there is a morphism of dendriform algebras $\mathit{End}(A) \to A$ given by

$$T \mapsto \sum_i T(u_i)v_i$$

Proof. □

Remark 4.15. There are in fact other, more primitive, dendriform structures on $\mathsf{End}(A)$ whenever A is an ϵ-bialgebra. These will be studied in future work.

5. Infinitesimal Hopf bimodules, pre-Lie bimodules, dendriform bimodules

In previous sections, we have shown how to construct a pre-Lie algebra from an ϵ-bialgebra and a dendriform algebra from a quasitriangular ϵ-bialgebra. These constructions are compatible, in the sense of (4.3). In this section we extend these constructions to the corresponding categories of bimodules.

The first step is to define the appropiate notion of bimodules over ϵ-bialgebras. Recall the notion of left infinitesimal Hopf modules from Definition 2.1. Right infinitesimal Hopf modules are defined similarly. We combine these two notions in the following:

Definition 5.1. Let (A, μ, Δ) be an ϵ-bialgebra. An infinitesimal Hopf bimodule (abbreviated ϵ-Hopf bimodule) over A is a space M endowed with maps

$$\lambda : A \otimes M \to M, \ \Lambda : M \to A \otimes M, \ \xi : M \otimes A \to M \ \text{and} \ \Xi : M \to M \otimes A$$

such that

(a) (M, λ, Λ) is a left ϵ-Hopf module over (A, μ, Δ),
(b) (M, ξ, Ξ) is a right ϵ-Hopf module over (A, μ, Δ),
(c) (M, λ, ξ) is a bimodule over (A, μ),
(d) (M, Λ, Ξ) is a bicomodule over (A, Δ), and
(e) the following diagrams commute:

$$\begin{array}{ccc} A \otimes M & \xrightarrow{id \otimes \Xi} & A \otimes M \otimes A \\ \lambda \downarrow & & \downarrow \lambda \otimes id \\ M & \xrightarrow[\Xi]{} & M \otimes A \end{array} \qquad \begin{array}{ccc} M \otimes A & \xrightarrow{\Lambda \otimes id} & A \otimes M \otimes A \\ \xi \downarrow & & \downarrow id \otimes \xi \\ M & \xrightarrow[\Lambda]{} & A \otimes M \end{array}$$

Example 5.2. For any ϵ-bialgebra (A, μ, Δ), the space $M = A \otimes A$ is an ϵ-Hopf bimodule via

$$\lambda = \mu \otimes id,\ \Lambda = \Delta \otimes id,\ \xi = id \otimes \mu \text{ and } \Xi = id \otimes \Delta\,.$$

Note that A itself, with the canonical bimodule and bicomodule structures, is not an ϵ-Hopf bimodule.

We will often use the following notation, for an ϵ-Hopf bimodule $(M, \lambda, \Lambda, \xi, \Xi)$:

$$\lambda(a \otimes m) = am, \quad \xi(m \otimes a) = ma, \quad \Lambda(m) = m_{-1} \otimes m_0 \text{ and } \Xi(m) = m_0 \otimes m_1\,. \tag{5.1}$$

As is well known, this notation efficiently encodes the bicomodule axioms. For instance, $m_{-2} \otimes m_{-1} \otimes m_0$ stands for $(\Delta \otimes id)\Lambda(m) = (id \otimes \Lambda)\Lambda(m)$, and $m_{-1} \otimes m_0 \otimes m_1$ for $(id \otimes \Xi)\Lambda(m) = (\Lambda \otimes id)\Xi(m)$.

Just as left ϵ-Hopf modules over A are left modules over $D(A)$ (Theorem 2.5), ϵ-Hopf bimodules over A are bimodules over $D(A)$.

Proposition 5.3. *Let A be a finite dimensional ϵ-bialgebra and M a space. If $(M, \lambda, \Lambda, \xi, \Xi)$ is an ϵ-Hopf bimodule over A as in* (5.1)*, then M is a bimodule over $D(A)$ via*

$$a \cdot m = am, \quad f \cdot m = f(m_{-1})m_0, \quad (a \bowtie f) \cdot m = f(m_{-1})am_0 \tag{5.2}$$

and

$$m \cdot a = ma, \quad m \cdot f = f(m_1)m_0, \quad m \cdot (a \bowtie f) = f(m_1)m_0 a\,. \tag{5.3}$$

Conversely, if M is a bimodule over $D(A)$ then M is an ϵ-Hopf bimodule over A and the structures are related as above.

Proof. Suppose M is an ϵ-Hopf bimodule over A. Then (M, λ, Λ) is a left ϵ-Hopf module and (M, ξ, Ξ) is a right ϵ-Hopf module over A. By Theorem 2.5, M is a left and right module over $D(A)$ by means of (5.2) and (5.3). It only remains to check that these structures commute.

By assumption (c) (resp. (d)) in Definition 5.1, the left action of A (resp. A') commutes with the right action of A (resp. A'). Similarly, by assumption (e), the left action of A (resp. A') commutes with the right action of A' (resp. A). Since, by Proposition 2.3, $D(A)$ is generated as an algebra by $A \oplus A'$, the previous facts guarantee that the left and right actions of $D(A)$ on M commute.

The converse is similar. □

Next, we will relate ϵ-Hopf bimodules over an ϵ-bialgebra A to bimodules over the associated pre-Lie algebra, and similarly for the case of a quasitriangular ϵ-bialgebra and the associated dendriform algebra. For this purpose, we recall the definition of bimodules over these types of algebras. There is a general notion from the theory of operads that dictates the bimodule axioms in each case [12]. In the cases of present interest, it turns out that the bimodule axioms are obtained from the axioms for the corresponding type of algebras by the following simple procedure. Each axiom for the given type of algebras yields three bimodule axioms, obtained by choosing one of the variables x, y or z and replacing it by a variable m from the bimodule (this may yield repeated axioms). This leads to the following definitions.

Definition 5.4. Let $(P, \circ)$ be a (left) pre-Lie algebra. A P-bimodule is a space M endowed with maps $P \otimes M \to M$, $x \otimes m \mapsto x \circ m$ and $M \otimes P \to M$, $m \otimes x \mapsto m \circ x$, such that

$$x \circ (y \circ m) - (x \circ y) \circ m = y \circ (x \circ m) - (y \circ x) \circ m\,, \tag{5.4}$$

$$x \circ (m \circ z) - (x \circ m) \circ z = m \circ (x \circ z) - (m \circ x) \circ z\,. \tag{5.5}$$

In Section 3 we encountered left P-modules (3.4). Note that any such can be turned into a P-bimodule by choosing the trivial right action $m \circ x \equiv 0$.

Definition 5.5. Let $(D, \succ, \prec)$ be a dendriform algebra. A D-bimodule is a vector space M together with four maps

$$\begin{array}{cccc} D\otimes M \to M & D\otimes M \to M & M\otimes D \to M & M\otimes D \to M \\ x\otimes m \mapsto x \succ m & x\otimes m \mapsto x \prec m & m\otimes x \mapsto m \succ x & m\otimes x \mapsto m \prec x \end{array}$$

such that

$$(x \prec y) \prec m = x \prec (y \prec m) + x \prec (y \succ m)\,,$$
$$x \succ (y \prec m) = (x \succ y) \prec m\,,$$
$$x \succ (y \succ m) = (x \prec y) \succ m + (x \succ y) \succ m\,,$$

$$(x \prec m) \prec z = x \prec (m \prec z) + x \prec (m \succ z)\,,$$
$$x \succ (m \prec z) = (x \succ m) \prec z\,,$$
$$x \succ (m \succ z) = (x \prec m) \succ z + (x \succ m) \succ z\,,$$

$$(m \prec y) \prec z = m \prec (y \prec z) + m \prec (y \succ z)\,,$$
$$m \succ (y \prec z) = (m \succ y) \prec z\,,$$
$$m \succ (y \succ z) = (m \prec y) \succ z + (m \succ y) \succ z\,.$$

It is easy to verify that if M is dendriform bimodule over D then it is also a pre-Lie bimodule over the associated pre-Lie algebra (4.2) by means of

$$x \circ m = x \succ m - m \prec x \quad \text{and} \quad m \circ x = m \succ x - x \prec m\,. \tag{5.6}$$

Next we show that the construction of pre-Lie algebras from ϵ-bialgebras can be extended to bimodules.

Proposition 5.6. *Let A be an ϵ-bialgebra and $(M, \lambda, \Lambda, \xi, \Xi)$ an ϵ-Hopf bimodule over A. Then M is a bimodule over the pre-Lie algebra $(A, \circ)$ of Theorem 3.2 via*

$$a \circ m = m_{-1}am_0 + m_0am_1 \quad \textit{and} \quad m \circ a = a_1ma_2\,. \tag{5.7}$$

Proof. Consider the first axiom in Definition 5.4. We have $b \circ m = m_{-1}bm_0 + m_0bm_1$. Using the axioms in Definition 5.1 we calculate

$$\Lambda(b \circ m) = m_{-2}\otimes m_{-1}bm_0 + m_{-1}b_1\otimes b_2m_0 + m_{-2}bm_{-1}\otimes m_0 + m_{-1}\otimes m_0bm_1\,,$$
$$\Xi(b \circ m) = m_{-1}bm_0\otimes m_1 + m_0b_1\otimes b_2m_1 + m_0bm_1\otimes m_2 + m_0\otimes m_1bm_2\,.$$

Hence,

$$a \circ (b \circ m) = m_{-2}am_{-1}bm_0 + m_{-1}b_1ab_2m_0 + m_{-2}bm_{-1}am_0 + m_{-1}am_0bm_1$$
$$+ m_{-1}bm_0am_1 + m_0b_1ab_2m_1 + m_0bm_1am_2 + m_0am_1bm_2\,.$$

On the other hand, $a \circ b = b_1ab_2$, so

$$(a \circ b) \circ m = m_{-1}b_1ab_2m_0 + m_0b_1ab_2m_1\,.$$

Therefore,

$$a \circ (b \circ m) - (a \circ b) \circ m = m_{-2}am_{-1}bm_0 + m_{-2}bm_{-1}am_0 + m_{-1}am_0bm_1$$
$$+ m_{-1}bm_0am_1 + m_0bm_1am_2 + m_0am_1bm_2\,.$$

Since this expression is symmetric under $a \leftrightarrow b$, Axiom (5.4) holds.

Consider now the second axiom. We have $m \circ b = b_1 m b_2$. Using the axioms in Definition 5.1 we calculate $\Lambda(m \circ b) = b_1 m_{-1} \otimes m_0 b_2 + b_1 \otimes b_2 m b_3$ and $\Xi(m \circ b) = b_1 m_0 \otimes m_1 b_2 + b_1 m b_2 \otimes b_3$. It follows that

$$a \circ (m \circ b) = b_1 m_{-1} a m_0 b_2 + b_1 a b_2 m b_3 + b_1 m_0 a m_1 b_2 + b_1 m b_2 a b_3 \,.$$

Since

$$(a \circ m) \circ b = b_1 m_{-1} a m_0 b_2 + b_1 m_0 a m_1 b_2 \,.$$

we deduce that

$$(*) \qquad a \circ (m \circ b) - (a \circ m) \circ b = b_1 a b_2 m b_3 + b_1 m b_2 a b_3 \,.$$

On the other hand, since $\Delta(a \circ b) = \Delta(b_1 a b_2) = b_1 \otimes b_2 a b_3 + b_1 a_1 \otimes a_2 b_2 + b_1 a b_2 \otimes b_3$, we have that

$$m \circ (a \circ b) = b_1 m b_2 a b_3 + b_1 a_1 m a_2 b_2 + b_1 a b_2 m b_3 \,,$$

and since

$$(m \circ a) \circ b = b_1 a_1 m a_2 b_2 \,,$$

we obtain that,

$$(**) \qquad m \circ (a \circ b) - (m \circ a) \circ b = b_1 m b_2 a b_3 + b_1 a b_2 m b_3 \,.$$

Comparing $(*)$ with $(**)$ with we see that Axiom (5.5) holds as well. □

Remark 5.7. Proposition 3.6 may be seen as the particular case of Proposition 5.6 when the right module and comodule structures on M are trivial (i.e., zero).

We set now to extend the construction of dendriform algebras from quasitriangular ϵ-bialgebras to the categories of bimodules. As in Section 4, it is convenient to consider the more general context of Baxter operators.

Definition 5.8. Let A be an associative algebra and $\beta_A : A \to A$ a Baxter operator (4.4). A Baxter operator on a A-bimodule M (relative to β_A) is a map $\beta_M : M \to M$ such that

$$\beta_A(a)\beta_M(m) = \beta_M\Big(a\beta_M(m) + \beta_A(a)m\Big) \,, \tag{5.8}$$

$$\beta_M(m)\beta_A(a) = \beta_M\Big(m\beta_A(a) + \beta_M(m)a\Big) \,. \tag{5.9}$$

Proposition 5.9. *Let A be an associative algebra, $\beta_A : A \to A$ a Baxter operator on A, M an A-bimodule and β_M a Baxter operator on M. Define new actions of A on M by*

$$a \succ m = \beta_A(a)m \,, \;\; m \succ a = \beta_M(m)a \,, \;\; a \prec m = a\beta_M(m) \;\; \text{and} \;\; m \prec a = m\beta_A(a) \,.$$

Equipped with actions, M is a bimodule over the dendriform algebra of Proposition 4.5.

Proof. Similar to the proof of Proposition 4.5. □

Proposition 5.10. *Let $r = \sum_i u_i \otimes v_i$ be a solution of the associative Yang-Baxter equation* (2.8) *in an associative algebra A. Let M be an A-bimodule. Then the map $\beta_M : M \to M$ defined by*

$$\beta_M(m) = \sum_i u_i m v_i$$

is a Baxter operator on M, relative to the Baxter operator on A of Proposition 4.3

Proof. Replacing the first tensor symbol in the associative Yang-Baxter equation (2.8) by a and the second by m one obtains (5.8). Replacing them in the other order yields (5.9). □

Finally, we have the desired construction of dendriform bimodules from bimodules over quasitriangular ϵ-bialgebras.

Corollary 5.11. *Let (A, r) be a quasitriangular ϵ-bialgebra, $r = \sum_i u_i \otimes v_i$. Let M be an arbitrary A-bimodule. Define new actions of A on M by*

$$a \succ m = \sum_i u_i a v_i m\,,\ m \succ a = \sum_i u_i m v_i a\,,\ a \prec m = \sum_i a u_i m v_i \ \text{ and } \ m \prec a = \sum_i m u_i a v_i\,. \tag{5.10}$$

Equipped with these actions, M is a bimodule over the dendriform algebra of Theorem 4.6.

Proof. Combine Propositions 5.10 and 5.9. □

We have thus constructed, from an ϵ-Hopf bimodule over an ϵ-bialgebra, a bimodule over the associated pre-Lie algebra (Proposition 5.6), and from a bimodule over a quasitriangular ϵ-bialgebra, a bimodule over the associated dendriform algebra (Corollary 5.11). Also, a bimodule over a dendriform algebra always yields a bimodule over the associated pre-Lie algebra (5.6). In order to close the circle, it remains to construct an ϵ-Hopf bimodule from a bimodule over a quasitriangular ϵ-bialgebra.

Proposition 5.12. *Let (A, r) be a quasitriangular ϵ-bialgebra and M an A-bimodule. Then M becomes a ϵ-Hopf bimodule over A via $\Lambda : M \to A \otimes M$ and $\Xi : M \to M \otimes A$ defined by*

$$\Lambda(m) = \sum_i u_i \otimes v_i m \quad \text{and} \quad \Xi(m) = -\sum_i m u_i \otimes v_i\,. \tag{5.11}$$

Proof. We already know, from Proposition 2.7, that Λ turns M into a left ϵ-Hopf module over A. Thus, axiom (a) in Definition 5.1 holds. Axiom (b) can be checked similarly: Ξ is coassociative because of the fact that $(id \otimes \Delta)(r) = -r_{13} r_{12}$, which holds according to [1, Proposition 5.5] (one must take into account Remark 2.6). Note that the minus sign in the definition of Ξ is essential, to ensure both this and the right ϵ-Hopf module axiom.

Axiom (c) holds by hypothesis. Axiom (d) holds because both $(id \otimes \Xi)\Lambda(m)$ and $(\Lambda \otimes id)\Xi(m)$ are equal to

$$-\sum_{i,j} u_i \otimes v_i m u_j \otimes v_i\,.$$

Axiom (e) holds because both $\Xi\lambda(a \otimes m)$ and $(\lambda \otimes id)(id \otimes \Xi)(a \otimes m)$ are equal to

$$-\sum_i a m u_i \otimes v_i\,,$$

and similarly for $\Lambda\xi$ and $(id \otimes \xi)(\Lambda \otimes id)$. This completes the proof. □

Let (A, r) be a quasitriangular ϵ-bialgebra. Consider the associated algebras as in diagram (4.3). In this section, we have constructed the corresponding diagram at the level of bimodules:

$$\begin{array}{ccc} \text{(Associative) bimodules} & \longrightarrow & \epsilon\text{-Hopf bimodules} \\ \downarrow & & \downarrow \\ \text{Dendriform bimodules} & \longrightarrow & \text{Pre-Lie bimodules} \end{array} \tag{5.12}$$

Each arrow is a functor, as morphisms are clearly preserved. The diagram is indeed commutative. Going around clockwise, we pass through the ϵ-Hopf bimodule with coactions $\Lambda(m) = \sum_i u_i \otimes v_i m$ and $\Xi(m) = -\sum_i m u_i \otimes v_i$, according to (5.11). The associated pre-Lie bimodule actions are, by (5.7),

$$a \circ m = \sum_i u_i a v_i m - \sum_i m u_i a v_i \quad \text{and} \quad m \circ a = \sum_i u_i m v_i a - \sum_i a u_i m v_i\,.$$

According to (5.10), these expressions are respectively equal to $a \succ m - m \prec a$ and $m \succ a - a \prec m$, which by (5.6) is the pre-Lie bimodule structure obtained by going counterclockwise around the diagram.

6. Brace algebras

In this section we explain how one may associate a brace algebra to an arbitrary ϵ-bialgebra, in a way that refines the pre-Lie algebra construction of Section 3 and that is compatible with the dendriform algebra construction of Section 4.

We provide the left version of the definition of brace algebras given in [6].

Definition 6.1. A (left) brace algebra is a space B equipped with multilinear operations $B^n \times B \to B$, $(x_1, \dots, x_n, z) \mapsto \langle x_1, \dots, x_n; z\rangle$, one for each $n \geq 0$, such that

$$\langle z\rangle = z$$

and for any n, $m \geq 1$,

$$\begin{aligned}(6.1)\qquad &\langle x_1, \dots, x_n; \langle y_1, \dots, y_m; z\rangle\rangle = \\ &\qquad \sum \langle X_0, \langle X_1; y_1\rangle, X_2, \langle X_3; y_2\rangle, X_4, \dots, X_{2m-2}, \langle X_{2m-1}; y_m\rangle, X_{2m}; z\rangle\,,\end{aligned}$$

where the sum takes place over all partitions of the ordered set $\{x_1, \dots, x_n\}$ into (possibly empty) consecutive intervals $X_0 \sqcup X_1 \sqcup \cdots \sqcup X_{2m}$.

The case $n = m = 1$ of Axiom 6.1 says

$$\langle x; \langle y; z\rangle\rangle = \langle x, y; z\rangle + \langle\langle x; y\rangle; z\rangle + \langle y, x; z\rangle\,. \tag{6.2}$$

The three terms on the right hand side correspond respectively to the partitions $(\{x\}, \emptyset, \emptyset)$, $(\emptyset, \{x\}, \emptyset)$ and $(\emptyset, \emptyset, \{x\})$.

The operation $x \circ y := \langle x; y\rangle$ endows B with a pre-Lie algebra structure. In fact, (6.2) shows that $x \circ (y \circ z) - (x \circ y) \circ z$ is symmetric under $x \leftrightarrow y$, so Axiom (3.1) holds. This defines a functor from brace algebras to pre-Lie algebras. The construction of pre-Lie algebras from ϵ-bialgebras in Section 3 can be refined accordingly, as we explain next.

In order to describe this refined construction, we must depart from our notational convention for coproducts and revert to Sweedler's original notation. Thus, in this section, the coproducts of an element b will be denoted by

$$\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$$

and the n-th iteration of the coproduct by

$$\Delta^{(n)}(b) = \sum_{(b)} b_{(1)} \otimes \cdots \otimes b_{(n+1)}\,.$$

Theorem 6.2. *Let A be an ϵ-bialgebra. Define operations $A^n \times A \to A$ by*

$$\langle a_1, \dots, a_n; b\rangle = \sum_{(b)} b_{(1)} a_1 b_{(2)} a_2 \dots b_{(n)} a_n b_{(n+1)}\,.$$

These operations turn A into a brace algebra.

Proof. The complete details of the proof will be provided elsewhere. The idea is simple: each term on the right hand side of (6.1) corresponds to a term in the expansion of

$$\Delta^{(n)}\Big(\sum_{(z)} z_{(1)} y_1 z_{(2)} y_2 \dots z_{(m)} y_m z_{(m+1)}\Big)$$

obtained by successive applications of (2.1). For instance, when $n = 2$ and $m = 1$, one has

$$\Delta^{(2)}\Big(\sum_{(z)} z_{(1)}yz_{(2)}\Big) = \sum_{(z)} z_{(1)}\otimes z_{(2)}\otimes z_{(3)}yz_{(4)} + z_{(1)}\otimes z_{(2)}yz_{(3)}\otimes z_{(4)} + z_{(1)}yz_{(2)}\otimes z_{(3)}\otimes z_{(4)}$$
$$+ \sum_{(z),(y)} z_{(1)}y_{(1)}\otimes y_{(2)}z_{(2)}\otimes z_{(3)} + z_{(1)}y_{(1)}\otimes y_{(2)}\otimes y_{(3)}z_{(2)} + z_{(1)}\otimes z_{(2)}y_{(1)}\otimes y_{(2)}z_{(3)}$$

Therefore,

$$\begin{aligned}\langle x_1, x_2; \langle y; z\rangle\rangle &= \sum_{(z)} z_{(1)}x_1z_{(2)}x_2z_{(3)}yz_{(4)} + z_{(1)}x_1z_{(2)}yz_{(3)}x_2z_{(4)} + z_{(1)}yz_{(2)}x_1z_{(3)}x_2z_{(4)}\\ &+ \sum_{(z),(y)} z_{(1)}y_{(1)}x_1y_{(2)}z_{(2)}x_2z_{(3)} + z_{(1)}y_{(1)}x_1y_{(2)}x_2y_{(3)}z_{(2)} + z_{(1)}x_1z_{(2)}y_{(1)}x_2y_{(2)}z_{(3)}\\ &= \langle x_1, x_2, y; z\rangle + \langle x_1, y, x_2; z\rangle + \langle y, x_1, x_2; z\rangle\\ &+ \langle\langle x_1; y\rangle, x_2; z\rangle + \langle\langle x_1, x_2; y\rangle; z\rangle + \langle x_1, \langle x_2; y\rangle; z\rangle\end{aligned}$$

which is Axiom (6.1). □

By construction, the first brace operation on A is simply

$$\langle a; b\rangle = \sum_{(b)} b_{(1)}ab_{(2)}\,,$$

which agrees with the pre-Lie operation (3.2). In this sense, the constructions of Theorems 3.2 and 6.2 are compatible.

Example 6.3. Consider the ϵ-bialgebra $k[\mathbf{x}, \mathbf{x}^{-1}]$ of divided differences (Examples 3.4). It is easy to see that for any $n \geq 0$ and $r \in \mathbb{Z}$,

$$\mu^{(n)}\Delta^{(n)}(\mathbf{x}^r) = \binom{r}{n}\mathbf{x}^{r-n}\,,$$

where it is understood, as usual, that $\binom{r}{n} = 0$ if $n > r \geq 0$ and $\binom{r}{n} = (-1)^n\binom{-r+n-1}{n}$ if $r < 0$. It follows that the brace algebra structure on $k[\mathbf{x}, \mathbf{x}^{-1}]$ is

$$\langle \mathbf{x}^{p_1}, \ldots, \mathbf{x}^{p_n}; \mathbf{x}^r\rangle = \binom{r}{n}\mathbf{x}^{r+p_1+\cdots+p_n-n}\,.$$

The brace axioms (6.1) boil down to a set of interesting identities involving binomial coefficients.

Frédéric Chapoton made us aware of the fact that if one applies the general construction of [15, Proposition 1] (dropping all signs) to the associative operad, one obtains precisely the brace subalgebra $k[\mathbf{x}]$ of our brace algebra $k[\mathbf{x}, \mathbf{x}^{-1}]$.

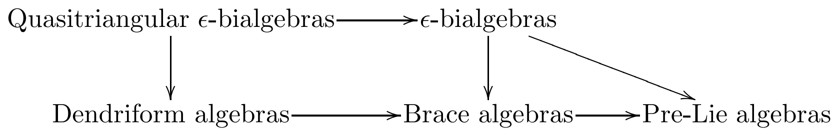
This example may be generalized in another direction. Namely, if A is a commutative algebra and $D : A \to A$ a derivation, then one obtains a brace algebra structure on A by defining

$$\langle x_1, \ldots, x_n; z\rangle = x_1 \cdots x_n \frac{D^n(z)}{n!}$$

(assuming $\mathsf{char}(k) = 0$). The example above corresponds to $A = k[\mathbf{x}, \mathbf{x}^{-1}]$, $D = \frac{d}{d\mathbf{x}}$.

Brace algebras sit between dendriform and pre-Lie algebras: Ronco has shown that one can associate a brace algebra to a dendriform algebra, by means of certain operations [26, Theorem 3.4]. Our constructions of dendriform and brace algebras from Theorems 4.6 and 6.2 are compatible with this functor.

In summary, one obtains a commutative diagram



The details will be provided elsewhere.

Appendix A. Infinitesimal bialgebras as comonoid objects

Ordinary bialgebras are bimonoid objects in the braided monoidal category of vector spaces, where the monoidal structure is the usual tensor product $V \otimes W$ and the braiding is the trivial symmetry $x \otimes y \mapsto y \otimes x$. In this appendix, we construct a certain monoidal category of algebras for which the comonoid objects are precisely ϵ-bialgebras. Related notions of bimonoid objects are discussed as well.

For the basics on monoidal categories the reader is referred to [24, Chapters VII and XI] and [19, Chapter XI]. The monoidal categories we consider possess a unit object, and whenever we refer to monoid objects these are assumed to be unital, even if not explicitly stated. Similarly, comonoid objects are assumed to be counital.

We start by recalling the well known *circle* tensor product of vector spaces.

Definition A.1. The circle product of two vector spaces V and W is

$$V \circ W = V \oplus W \oplus (V \otimes W)\,.$$

We denote the elements of this space by triples $(v, w, x \otimes y)$. The circle product of maps $f : V \to X$ and $g : W \to Y$ is

$$(f \circ g)(v, w, x \otimes y) = \big(f(v), g(w), (f \otimes g)(x \otimes y)\big)\,.$$

Both spaces $(U \circ V) \circ W$ and $U \circ (V \circ W)$ can be canonically identified with

$$U \oplus V \oplus W \oplus (U \otimes V) \oplus (U \otimes W) \oplus (V \otimes W) \oplus (U \otimes V \otimes W)\,.$$

This gives rise to a natural isomorphism $(U \circ V) \circ W \cong U \circ (V \circ W)$ which satisfies the pentagon for associativity. This endows the category of vector spaces with a monoidal structure, for which the unit object is the zero space. We denote this monoidal category by $(\mathsf{Vec}, \circ, 0)$.

Let $(\mathsf{Vec}, \otimes, k)$ denote the usual monoidal category of vector spaces, where the monoid objects are unital associative algebras and the comonoid objects are counital coassociative coalgebras. There is an obvious monoidal functor $\alpha : (\mathsf{Vec}, \circ, 0) \to (\mathsf{Vec}, \otimes, k)$ defined by

$$V \mapsto V \oplus k\,.$$

It is the so called *augmentation* functor.

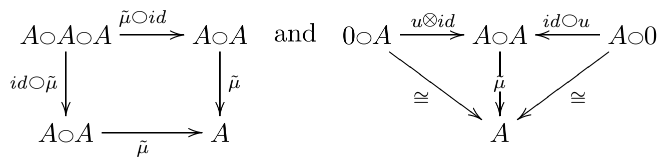
Monoids and comonoids in $(\mathsf{Vec}, \circ, 0)$ are easy to describe: they are, respectively, non unital algebras and non counital coalgebras (Proposition A.2, below). Monoids and comonoids are preserved by monoidal functors. In the present situation this simply says that a non unital algebra can be canonically augmented into a unital algebra, and similarly for coalgebras.

Proposition A.2. *A unital monoid object in $(\mathsf{Vec}, \circ, 0)$ is precisely an associative algebra, not necessarily unital. A counital comonoid object is precisely a coassociative coalgebra, not necessarily counital.*

Proof. Let (A, μ) be an associative algebra, $\mu(a \otimes a') = aa'$. Define a map $\tilde{\mu} : A \circ A \to A$ by

$$(a, a', x \otimes x') \mapsto a + a' + xx'\,. \tag{A.1}$$

Let $u : 0 \to A$ be the unique map. Then, the diagrams



commute. Thus, $(A, \tilde{\mu}, u)$ is a unital monoid in $(\mathsf{Vec}, \bigcirc, 0)$.

Conversely, if $(A, \tilde{\mu}, u)$ is a unital monoid in $(\mathsf{Vec}, \bigcirc, 0)$, then $\tilde{\mu}$ must be of the form (A.1) for an associative multiplication μ on A, by the commutativity of the diagrams above.

The assertion for comonoids is similar. The comultiplication $\Delta : A \to A \otimes A$ is related to the comonoid structure $\tilde{\Delta} : A \to A \bigcirc A$ by

$$\tilde{\Delta}(a) = (a, a, \Delta(a)) \tag{A.2}$$

and $\epsilon : A \to 0$ is the unique map. $\square$

Remark A.3. It is natural to wonder if there is a braiding on the monoidal category $(\mathsf{Vec}, \bigcirc, 0)$ for which the bimonoid objects are precisely ϵ-bialgebras. We know of two braidings on $(\mathsf{Vec}, \bigcirc, 0)$. The corresponding notions of bimonoid objects are briefly discussed next. Neither yields ϵ-bialgebras.

(1) For any spaces V and W, consider the map $\sigma_{V,W} : V \bigcirc W \to W \bigcirc V$ defined by

$$(v, w, x \otimes y) \mapsto (w, v, y \otimes x) .$$

This family of maps clearly satisfies the axioms for a braiding on the monoidal category $(\mathsf{Vec}, \bigcirc, 0)$. Under the monoidal functor α, the braiding σ corresponds to the usual braiding on $(\mathsf{Vec}, \otimes, k)$ (the trivial symmetry). For this reason, a bimonoid object in $(\mathsf{Vec}, \bigcirc, 0, \sigma)$ can be canonically augmented into an ordinary bialgebra.

It follows from Proposition A.2) that a bimonoid object in $(\mathsf{Vec}, \bigcirc, 0, \sigma)$ is a space A, equipped with an associative algebra structure $A \otimes A \to A$, $a \otimes a' \mapsto aa'$, and a coassociative coalgebra structure $\Delta : A \to A \otimes A$, $a \mapsto a_1 \otimes a_2$, related by the axiom

$$\Delta(aa') = a \otimes a' + a' \otimes a + aa'_1 \otimes a'_2 + a'_1 \otimes aa'_2 + a_1 a' \otimes a_2 + a_1 \otimes a_2 a' + a_1 a'_1 \otimes a_2 a'_2 .$$

This is *not* the axiom which defines ϵ-bialgebras (2.1).

The axiom above is a translation of the fact that the map $A \to A \bigcirc A$ must be a morphism of monoids. We omit this calculation, but provide an explicit description of the monoid structure on $A \bigcirc A$. More generally, we describe the tensor product of two monoids A and B in $(\mathsf{Vec}, \bigcirc, 0, \sigma)$.

According to Proposition A.2, the monoid structure on $A \bigcirc B$ is uniquely determined by an associative multiplication on the space $A \bigcirc B$. We describe this multiplication, in terms of those of A and B. It is

$$(a, b, x \otimes y) \cdot (a', b', x' \otimes y') = (aa', bb', a \otimes b' + a' \otimes b + ax' \otimes y' + x' \otimes by' + xa' \otimes y + x \otimes yb' + xy \otimes x'y') .$$

This is the result of composing

$$(A \bigcirc B) \otimes (A \bigcirc B) \hookrightarrow (A \bigcirc B) \bigcirc (A \bigcirc B) \xrightarrow{id \bigcirc \sigma_{B,A} \bigcirc id} (A \bigcirc A) \bigcirc (B \bigcirc B) \xrightarrow{\tilde{\mu}_A \bigcirc \tilde{\mu}_B} A \bigcirc B .$$

(2) There exists a second braiding on the monoidal category $(\mathsf{Vec}, \bigcirc, 0)$, for which bimonoid objects are somewhat closer to ϵ-bialgebras. It is the family of maps $\beta_{V,W} : V \bigcirc W \to W \bigcirc V$ defined by

$$(v, w, x \otimes y) \mapsto (w, v, 0) .$$

Apart from the fact that β is *not* an isomorphism, the braiding axioms are satisfied by β. This allows us to construct a monoid structure on the circle product of two monoids in $(\mathsf{Vec}, \circ, 0)$, and therefore to speak of bimonoid objects in $(\mathsf{Vec}, \circ, 0, \beta)$, as usual.

Since the monoidal functor α does not preserve this braiding, the augmentation of a bimonoid in $(\mathsf{Vec}, \circ, 0, \beta)$ is not an ordinary bialgebra. Neither is it true that these bimonoid objects are ϵ-bialgebras. In fact, a bimonoid object in $(\mathsf{Vec}, \circ, 0, \beta)$ is a space A, equipped with an associative algebra structure and a coassociative coalgebra structure, as above, related by the axiom

$$\Delta(aa') = a \otimes a' + aa'_1 \otimes a'_2 + a_1 \otimes a_2 a' \,.$$

Compare with Axiom (2.1) for infinitesimal bialgebras.

This can be deduced from the following description of the tensor product in $(\mathsf{Vec}, \circ, 0, \beta)$ of two monoid objects A and B. This structure is determined by the following (associative) multiplication on $A \circ B$:

$$(a, b, x \otimes y) \cdot (a', b', x' \otimes y') = (aa', bb', a \otimes b' + ax' \otimes y' + x \otimes yb') \,.$$

This is the result of composing

$$(A \circ B) \otimes (A \circ B) \hookrightarrow (A \circ B) \circ (A \circ B) \xrightarrow{id \circ \beta_{B,A} \circ id} (A \circ A) \circ (B \circ B) \xrightarrow{\tilde{\mu}_A \circ \tilde{\mu}_B} A \circ B \,.$$

Let Alg denote the category of monoids in $(\mathsf{Vec}, \circ, 0)$, that is, associative algebras which are not necessarily unital. We define a new monoidal structure on this category, independent of any braiding on $(\mathsf{Vec}, \circ, 0)$. We will show that ϵ-bialgebras are precisely comonoid objects in the resulting monoidal category.

Proposition A.4. *Let A and B be associative algebras, not necessarily unital. Then $A \circ B$ is an associative algebra via*

$$(a, b, x \otimes y) \cdot (a', b', x' \otimes y') = (aa', bb', ax' \otimes y' + x \otimes yb') \,. \tag{A.3}$$

Proof. Consider the algebra $R = A \oplus B$ and the R-bimodule $M = A \otimes B$, with

$$(a, b) \cdot x \otimes y = ax \otimes y \quad \text{and} \quad x \otimes y \cdot (a, b) = x \otimes yb \,.$$

The algebra $A \circ B$ is precisely the trivial extension $R \oplus M$, where the multiplication is

$$(r, m) \cdot (r', m') = (rr', rm' + mr') \,.$$

□

If $f : A \to A'$ and $g : B \to B'$ are morphisms of algebras, then so is $f \circ g : A \circ B \to A' \circ B'$. In this way, $(\mathsf{Alg}, \circ, 0)$ becomes a monoidal category.

Proposition A.5. *A counital comonoid object in the monoidal category $(\mathsf{Alg}, \circ, 0)$ is precisely an ϵ-bialgebra.*

Proof. Let (A, μ, Δ) be an ϵ-bialgebra. By Proposition A.2, A may be seen as a monoid and comonoid in $(\mathsf{Alg}, \circ, 0)$. It only remains to verify that the comonoid structure $\tilde{\Delta} : A \to A \circ A$, $\tilde{\Delta}(a) = (a, a, \Delta(a))$, is a morphism of algebras. This is clear from (A.3) and (2.1).

The converse is similar. □

The category of modules over an ordinary bialgebra H is monoidal: the tensor product of two H-modules acquires an H-module structure by restricting the natural $H \otimes H$-module structure via the comultiplication $\Delta : H \to H \otimes H$. There is no analogous construction for arbitrary ϵ-bialgebras. However, it is possible to construct tensor products of certain modules over ϵ-bialgebras, as discussed next.

Proposition A.6. *Let A and B be associative algebras. Let M be a right A-module and N a left B-module. Then $A\bigcirc N$ is a left $A\bigcirc B$-module via*

$$(a, b, x\otimes y) \cdot (a', n, x'\otimes v) := (aa', bn, ax'\otimes v + x\otimes yn)\,,$$

and $M\bigcirc B$ is a right $A\bigcirc B$-module via

$$(m, b', u\otimes y') \cdot (a, b, x\otimes y) := (ma, b'b, ub\otimes y' + mx\otimes y)\,.$$

Proof. Similar to the proof of Proposition A.4. □

Let A be an ϵ-bialgebra and N a left A-module. It is possible to define a left A-module structure on $A\bigcirc N$, by restricting the structure of Proposition A.6 along the morphism of algebras $\tilde{\Delta} : A \to A\bigcirc A$. The resulting action of A on $A\bigcirc N$ is

$$a \cdot (a', n, x\otimes v) = (aa', an, ax\otimes v + a_1\otimes a_2 n)\,. \tag{A.4}$$

Appendix B. Counital infinitesimal bialgebras

Definition B.1. An ϵ-bialgebra (A, μ, Δ) is said to be counital if the underlying coalgebra is counital, that is, if there exists a map $\eta : A \to k$ such that $(id\otimes\eta)\Delta = id = (\eta\otimes id)\Delta$.

The map η is necessarily unique and is called the counit of A. We use η instead of the customary ε to avoid confusion with the abbreviation for infinitesimal bialgebras.

Recall that if an ϵ-bialgebra A is both unital *and* counital then $A = 0$ [1, Remark 2.2]. Nevertheless, many ϵ-bialgebras arising in practice are either unital or counital. In this appendix we study counital ϵ-bialgebras; all constructions and results admit a dual version that applies to unital ϵ-bialgebras.

We first show that counital ϵ-bialgebras can be seen as comonoid objects in a certain monoidal category of algebras. This construction is parallel to that for arbitrary ϵ-bialgebras discussed in Appendix A. The two constructions are related by means of a pair of monoidal functors, but neither is more general than the other.

Lemma B.2. *Let A be a counital ϵ-bialgebra with counit η. Then*

$$\eta(aa') = 0 \textit{ for all } a,\, a' \in A.$$

Proof. We show that any coderivation $D : M \to C$ from a counital bicomodule (M, s, t) to a counital coalgebra (C, Δ, η) maps to the kernel of η. The result follows by applying this remark to the coderivation $\mu : A\otimes A \to A$.

We have

$$\begin{aligned}
\eta D = (\eta\otimes\eta)\Delta D &= (\eta\otimes\eta)\big((id_C\otimes D)t + (D\otimes id_C)s\big)\\
&= (id_k\otimes\eta D)(\eta\otimes id_M)t + (\eta D\otimes id_k)(id_M\otimes\eta)s\\
&= \eta D + \eta D = 2\cdot\eta D,
\end{aligned}$$

whence $\eta D = 0$. □

This motivates the following definition.

Definition B.3. Let (A, μ) be an algebra over k, not necessarily unital. We say that it is *augmented* if there is given a map $\eta : A \to k$ such that

$$\eta(aa') = 0 \text{ for all } a,\, a' \in A.$$

A morphism between augmented algebras (A, η_A) and (B, η_B) is a morphism of algebras $f : A \to B$ such that $\eta_B f = \eta_A$.

Proposition B.4. *Let* (A, η_A) *and* (B, η_B) *be augmented algebras. Then* $A \otimes B$ *is an associative algebra with multiplication*

$$(a \otimes b) \cdot (a' \otimes b') := \eta_B(b) aa' \otimes b' + \eta_A(a') a \otimes bb' \,. \tag{B.1}$$

Moreover, $A \otimes B$ *is augmented by*

$$\eta_{A \otimes B}(a \otimes b) := \eta_A(a) \eta_B(b) \,.$$

Proof. The first assertion is Lemma 3.5.b in [1] and the second is straightforward. □

We denote the resulting augmented algebra by $A \otimes_\epsilon B$. This operation defines a monoidal structure on the category of augmented algebras over k. The unit object is the base field k equipped with the zero multiplication and the identity augmentation. We denote this monoidal category by $(\mathsf{AAlg}, \otimes_\epsilon, k)$.

Proposition B.5. *A counital comonoid object in the monoidal category* $(\mathsf{AAlg}, \otimes_\epsilon, k)$ *is precisely a counital* ϵ*-bialgebra.*

Proof. Start from a counital ϵ-bialgebra (A, μ, Δ, η). By Lemma B.2, (A, η) is an augmented algebra. Moreover, by Lemma 3.6.b in [1], $\Delta : A \to A \otimes_\epsilon A$ is a morphism of algebras, and it preserves the augmentations by counitality. Clearly, $\eta : A \to k$ is also a morphism of augmented algebras. Thus, (A, μ, Δ, η) is a counital comonoid in $(\mathsf{Alg}, \otimes_\epsilon, k)$.

Conversely, let A be a counital comonoid in $(\mathsf{AAlg}, \otimes_\epsilon, k)$. First of all, the counit $A \to k$ must preserve the augmentations of A and k, so it must coincide with the augmentation of A. The comultiplication must be a morphism of algebras $A \to A \otimes_\epsilon A$. This implies Axiom (2.1), by definition of the algebra structure on $A \otimes_\epsilon A$ and counitality. Thus A is a counital ϵ-bialgebra. □

Remark B.6. An augmented algebra may be seen as a monoid in a certain monoidal category of "augmented vector spaces". However, the monoidal structure on $(\mathsf{AAlg}, \otimes_\epsilon, k)$ does not come from a braiding on the larger category of augmented vector spaces. For this reason, we cannot view counital ϵ-bialgebras as bimonoid objects. The situation parallels that encountered in Appendix A for arbitrary ϵ-bialgebras. In fact, there is pair of monoidal functors relating the two situations, as we discuss next.

Remark B.7. Given a non unital algebra A, let $A^+ := A \oplus k$, with algebra structure

$$(a, x) \cdot (b, y) = (ab, 0) \,. \tag{B.2}$$

Note that A^+ is not the usual augmentation of A; in fact, A^+ is non unital. Define $\eta(a, x) = x$. Then $\eta((a, x) \cdot (b, y)) = 0$, so (A^+, η) is an augmented algebra in the sense of Definition B.3. Moreover, it is easy to see that there is a natural isomorphism of augmented algebras

$$(A \circ B)^+ \cong A^+ \otimes_\epsilon B^+ \,.$$

The application $A \mapsto A^+$ is thus a monoidal functor

$$(\mathsf{Alg}, \circ, 0) \to (\mathsf{AAlg}, \otimes_\epsilon, k) \,.$$

The fact that comonoid objects are preserved by this monoidal functor simply says that any ϵ-bialgebra A can be made into a counital ϵ-bialgebra A^+, by extending the comultiplication via $\Delta(1) = 1 \otimes 1$ and the multiplication as in (B.2).

In the other direction, consider the forgetful functor

$$(\mathsf{AAlg}, \otimes_\epsilon, k) \to (\mathsf{Alg}, \circ, 0), \quad (A, \eta) \mapsto A \,.$$

It is easy to see that the map

$$A \otimes_\epsilon B \to A \circ B, \quad a \otimes b \mapsto \eta_B(b) a + \eta_A(a) b + a \otimes b \tag{B.3}$$

is a (natural) morphism of algebras. It follows that the forgetful functor is *lax monoidal*. The fact that comonoid objects are preserved by this type of functors simply says in this case that any counital ϵ-bialgebra is in particular an ϵ-bialgebra.

Neither functor between these two categories of algebras is a monoidal equivalence. For this reason, neither situation in Appendices A and B is more general than the other.

The following is the analog of Proposition A.6 for augmented algebras.

Proposition B.8. *Let A and B be augmented algebras. Let M be a right A-module and N a left B-module. Then $A{\otimes}N$ is a left $A{\otimes}_\epsilon B$-module via*

$$(a{\otimes}b)\cdot(a'{\otimes}n) := \eta_B(b)aa'{\otimes}n + \eta_A(a')a{\otimes}bn$$

and $M{\otimes}B$ is a right $A{\otimes}_\epsilon B$-module via

$$(m{\otimes}b')\cdot(a{\otimes}b) := \eta_A(a)m{\otimes}b'b + \eta_B(b')ma{\otimes}b\,.$$

Proof. Similar to the proof of Proposition B.4. □

One may similarly show that the map

$$A{\otimes}N \to A{\bigcirc}N, \quad a{\otimes}n \mapsto \eta_A(a)n + a{\otimes}n$$

is a morphism of left $A{\otimes}_\epsilon B$-modules, where $A{\bigcirc}N$ is viewed as a left $A{\otimes}_\epsilon B$-module by restriction via the morphism of algebras (B.3).

Finally, we discuss the analog of the construction (A.4) for counital ϵ-bialgebras, and apply these general considerations to the construction of an ϵ-Hopf module.

Let A be a counital ϵ-bialgebra and N a left A-module. It is possible to define a left A-module structure on $A{\otimes}N$, by restricting the structure of Proposition B.8 along the morphism of augmented algebras $\Delta : A \to A{\otimes}_\epsilon A$. By counitality, the action of A on $A{\otimes}N$ reduces to

$$a\cdot(a'{\otimes}n) = aa'{\otimes}n + \eta(a')a_1{\otimes}a_2 n\,. \tag{B.4}$$

We denote this module structure on the space $A{\otimes}N$ by $A{\otimes}_\epsilon N$.

Our next result describes ϵ-Hopf modules over counital ϵ-bialgebras in a way that is analogous to the definition of Hopf modules over an ordinary Hopf algebra.

Recall that a left Hopf module over a Hopf algebra H is a space M that is both a left module and comodule over H and for which the comodule structure map $M \to H{\otimes}M$ is a morphism of left H-modules [25, Definition 1.9.1]. It is understood that $H{\otimes}M$ is a left H-module by restriction via the comultiplication of H.

Proposition B.9. *Let A be a counital ϵ-bialgebra. Let $\lambda : A{\otimes}N \to N$ be a left A-module structure on N and $\Lambda : N \to A{\otimes}N$ a counital comodule structure on N. Then (N, λ, Λ) is an ϵ-Hopf module over A if and only if $\Lambda : N \to A{\otimes}_\epsilon N$ is a morphism of left A-modules.*

Proof. Write $\lambda(a{\otimes}n) = an$ and $\Lambda(n) = n_{-1}{\otimes}n_0$. According to (B.4),

$$\begin{aligned} a\cdot\Lambda(n) &= an_{-1}{\otimes}n_0 + \eta_A(n_{-1})\,a_1{\otimes}a_2 n_0 \\ &= an_{-1}{\otimes}n_0 + a_1{\otimes}a_2 n\,, \end{aligned}$$

by counitality for N. Thus, Λ is a morphism of A-modules if and only if

$$\Lambda(an) = an_{-1}{\otimes}n_0 + a_1{\otimes}a_2 n\,,$$

which is Axiom (2.2) in the definition of ϵ-Hopf module. □

Next, we make use of Proposition B.9 to obtain the general construction of ϵ-Hopf modules of Example 2.2.3.

First, note that the tensor product construction of Proposition B.8 is associative, in the sense that if A, B and C are augmented algebras and N is a left C-module, then

$$(A{\otimes}_\epsilon B){\otimes}_\epsilon N \cong A{\otimes}_\epsilon(B{\otimes}_\epsilon N)$$

as left $A \otimes_\epsilon B \otimes_\epsilon C$-modules. In fact, one has

$$\big((a \otimes b) \otimes c\big) \cdot \big((a' \otimes b') \otimes n\big) = \eta_C(c)\eta_B(b) aa' \otimes b' \otimes n + \eta_C(c)\eta_A(a') a \otimes bb' \otimes n + \eta_A(a')\eta_B(b') a \otimes b \otimes cn = \big(a \otimes (b \otimes c)\big) \cdot \big(a' \otimes (b' \otimes n)\big)\,.$$

On the other hand, if $f : A \to B$ is a morphism of augmented algebras and N is a left C-module, then $f \otimes id_N : A \otimes_\epsilon N \to B \otimes_\epsilon N$ is a morphism of left $A \otimes_\epsilon C$-modules, where $B \otimes_\epsilon N$ is an $A \otimes_\epsilon C$-module by restriction via the morphism of algebras $f \otimes id_C : A \otimes_\epsilon C \to B \otimes_\epsilon C$.

Now let us apply these considerations to a left module N over a counital ϵ-bialgebra A, $B = A \otimes_\epsilon A$, $C = A$ and $f = \Delta$. We obtain that

$$\Delta \otimes id_N : A \otimes_\epsilon N \to A \otimes_\epsilon A \otimes_\epsilon N$$

is a morphism of left $A \otimes_\epsilon A$-modules. Hence, it is also a morphism of left A-modules, by restriction via Δ. An application of Proposition B.9 then yields the following

Corollary B.10. *Let A be a counital ϵ-bialgebra and N a left N-module. Let $M = A \otimes_\epsilon N$, an A-module as in* (B.4). *Define $\Lambda : M \to A \otimes M$ by*

$$\Lambda(a \otimes n) = a_1 \otimes a_2 \otimes n\,.$$

With these module and comodule structures, M is a left ϵ-Hopf module over A.

In this paper, quasitriangular ϵ-bialgebras play an important role (Section 4). Our last result shows that the classes of counital ϵ-bialgebras and quasitriangular ϵ-bialgebras are disjoint.

Proposition B.11. *If a quasitriangular ϵ-bialgebra A is counital then $A = 0$.*

Proof. Let $r = \sum u_i \otimes v_i$ be the canonical element and η the counit. According to (2.9), we have

$$\Delta(a) = \sum_i u_i \otimes v_i a - a u_i \otimes v_i\,.$$

Applying $\eta \otimes id$ and using Lemma B.2 we deduce

$$a = \sum_i \eta(u_i) v_i a \text{ for every } a \in A\,.$$

Similarly, applying $id \otimes \eta$ we deduce

$$a = -\sum_i a u_i \eta(v_i) \text{ for every } a \in A\,.$$

Thus, A has a left unit and a right unit. These must therefore coincide and A must be unital. But an ϵ-bialgebra A that is both unital and counital must be 0 [1, Remark 2.2]. □

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Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: maguiar@math.tamu.edu
URL: http://www.math.tamu.edu/~maguiar