An introduction to algebraic K-theory

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The purpose of this talk is to give a brief introduction to algebraic K-theory in the sense of Quillen. Since defining the higher K-groups was an effort of over a decade of work, it seems worthwhile to begin by mentioning the initial motivation and constructions.

1 Lower K-groups of a ring

The story starts within algebraic geometry, when in 1957 Grothendieck defined K_0 of an algebraic variety (which we now call the Grothendieck group of a variety) in order to prove a generalization of the Riemann-Roch theorem. It was defined as follows:

Definition 1.1. Let X be an algebraic variety, and consider the category $\mathcal{P}(X)$ of vector bundles over X of finite rank. The Grothendieck group $K_0(X)$ is defined as the abelian group given by:

- generators: one symbol $[E]$ for every isomorphism class in $\mathcal{P}(X)$,
- relations: $[E_2] = [E_1] + [E_3]$ for every short exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

Motivated by Grothendieck's construction, Atiyah and Hirzebruch (1959-1961) defined K^0 of a topological space by using topological vector bundles, and showed that, in fact, the construction could be extended to higher groups K^n that assemble into a cohomology theory K^*.

It was then hoped that one could analogously extend K_0 back in the algebraic setting, at least for the case of an affine scheme $X = Spec(R)$, since in that case, vector bundles over X of finite rank correspond to finitely generated projective R-modules; one could thus drop the geometry and work in this purely algebraic setting. Rewriting Grothendieck's original definition in terms of rings and their modules, we get the definition of K_0 of a ring:

Definition 1.2. Let R be a ring, and consider the category $\mathcal{P}(R)$ of finitely generated projective R-modules. The Grothendieck group $K_0(R)$ is defined as the abelian group given by:

- generators: one symbol $[P]$ for every isomorphism class in $\mathcal{P}(R)$,
- relations: $[P_2] = [P_1] + [P_3]$ for every short exact sequence

$$0 \to P_1 \to P_2 \to P_3 \to 0.$$

Just in case it's not immediate from the definition, let's clarify how addition is defined for the generators. Given two classes $[P_1], [P_2]$, we can always consider the trivial extension

$$0 \to P_1 \to P_1 \oplus P_2 \to P_2 \to 0,$$

and thus the relation induced by this exact sequence gives

$$[P_1] + [P_2] = [P_1 \oplus P_2].$$

Then, one can see than an alternate description of $K_0(R)$ can be obtained by considering the monoid formed by these iso classes, together with the addition induced by direct sum; then take the group completion of that monoid, adding the necessary formal inverses to get a group, and lastly mod out by any remaining relations coming from short exact sequences.[1]

The hope was to be able to get, just like in the topological case, a sequence of abelian groups $K_0, K_1, \ldots, K_n, \ldots$ starting with the Grothendieck group, that produced long exact sequences in some meaningful way.

The algebraic version proved much harder than its topological counterpart, with the next K-group of a ring appearing in 1964 due to Bass: he defines

$$K_1(R) = GL(R)/[GL(R), GL(R)]$$

where $GL(R)$ denotes the infinite general linear group, and the brackets its commutator. Evidence of the correctness of this definition is given, among other things, by the fact that there exists an exact sequence

$$K_1(\oplus_{\mathfrak{p}} R/\mathfrak{p}) \to K_1(R) \to K_1(F) \to K_0(\oplus_{\mathfrak{p}} R/\mathfrak{p}) \to K_0(R) \to K_0(F) \to 0 \tag{1}$$

where R is a Dedekind domain, F its field of fractions and $\mathfrak{p}$ runs over its prime ideals. Later, in 1967, Milnor proposes a definition of $K_2(R)$ that succeeds in extending the exact sequence above by two terms.

Before moving on to the higher K-groups, I should briefly try to give indications of why this is worthwhile. For the categorically inclined, the evidence I've given so far is quite possibly compelling enough to justify the search for an extension to higher invariants. But this was not the only reason behind this project: it was quickly seen that the groups K_0 and K_1

[1] In fact, since every short exact sequence ending in a projective splits, there will be no extra relations, but this general description will hold in the more general cases we will deal with in a bit.

carry relevant information about the ring. To name a few simple examples, $K_0(R) = \mathbb{Z} \oplus Cl(R)$ computes the class group of a ring when R is a Dedekind domain[2], and $K_1(\mathbb{F}) = \mathbb{F}^\times$ for any field. Therefore, it was to be expected that higher invariants would contain other valuable information that would help further our understanding of these structures.

2 Higher K-theory: from rings to abelian categories

In the early 70's, various definitions of the higher K-groups of a ring were proposed, most of which were later shown to agree. The most popular of these is probably Quillen's "+" construction, but there were other constructions due to Swan, and Karoubi-Villamayor, to name a few.

Even though computational tools were developed for each, they all had the drawback of not producing the desired long exact sequences, or other nice foundational results that are present for the Q-construction.

The most notable development came in 1972-1973, when Quillen defined the higher K-groups for a class of structures more general than rings, and then proceeded to extend the fundamental results present for K_0, K_1 and K_2 (among them, the Localization theorem providing a long exact sequence that completes 1).

In what remains of this talk, we will define Quillen's construction for the higher K-groups, as well as the class of structures it applies to.

2.1 Quillen's Q-construction

Let's take a moment to go back to the definition of $K_0(R)$. We can see that the construction of the Grothendieck group does not make use of the full structure of the category $\mathcal{P}(R)$ of finitely generated projective modules per se, but rather of the fact that this category has a notion of short exact sequence. These are used to extract the information of how our building blocks $[P]$ "split", or equivalently, of how smaller, more elemental building blocks $[P_i]$ assemble to form larger building blocks.

Based on the above, one could venture that in order to define K_0, all we really need is a category with an appropriate notion of "short exact sequence". Those of you interested in homological algebra will surely know that this setting is that of abelian categories. There, the usual notions of kernel, cokernel and image are well-behaved, which allows us to define short exact sequences just like we do for R-Mod.[3] If you're not familiar with

[2] In case you don't recall the exact definition, the class group of a ring tells you how far the ring is from having unique prime factorizations.

[3] Actually, Quillen defines the Q-construction in the setting of exact categories. These are subcategories of abelian categories where one can customize the class of short exact sequences available in the category.

abelian categories, you will not be far off by thinking about R-Mod.

Given an abelian category $\mathcal{A}$, we can then define $K_0(\mathcal{A})$ by mimicking the definition of the Grothendieck group of a ring given in 1.2.

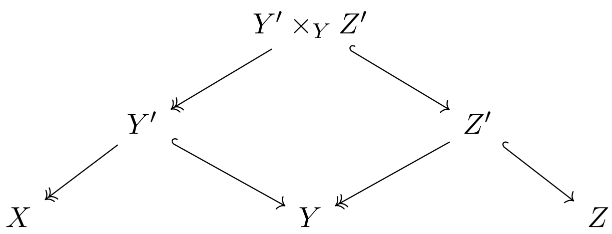
In order to construct the K-theory groups of an abelian category $\mathcal{A}$, Quillen defines an auxiliary category $\mathcal{Q}\mathcal{A}$, as follows.

Definition 2.1. Given an abelian category $\mathcal{A}$, $\mathcal{Q}\mathcal{A}$ is the category with the same objects as $\mathcal{A}$, and for which a morphism from X to Y is an equivalence class of diagrams of the form

$$X \overset{p}{\twoheadleftarrow} Y' \overset{i}{\hookrightarrow} Y$$

where i is a monomorphism and p a epimorphism in $\mathcal{A}$. Two such diagrams are equivalent if there is an isomorphism between them which is the identity on X and Y.

Composition is given by pullback:



With this category in hand, one can define the K-theory space of an abelian category $\mathcal{A}$ by

$$K\mathcal{A} = \Omega|N(\mathcal{Q}\mathcal{A})| = \Omega B\mathcal{Q}\mathcal{A},$$

and then the K-groups of $\mathcal{A}$ are

$$K_n(\mathcal{A}) = \pi_n K\mathcal{A} = \pi_{n+1}|N(\mathcal{Q}\mathcal{A})|.$$

With this approach, the K-groups of a ring R are precisely the K-groups of the category of finitely generated projective R-modules.[4]

At this point it may seem like we're requiring a huge leap of faith: we started with very concrete definitions for K_0 and K_1 of rings, and ended up with this seemingly untractable abstract definition of higher K-groups. One could indeed show that the newly defined K_0 and K_1 agree with the desired groups, and that, for example, there exist key theorems (called Localization and Devissage) that yield a long exact sequence that continues (1). For more on these, come to Inna's talk!

In an attempt to bring things back down to Earth, we conclude this talk by giving some indication of why this is in fact an extension of the notions we had defined previously: we show that this construction yields the correct K_0.

[4] We're slightly lying here, since that category is not abelian, but exact. The definitions are exactly the same, though.

Proposition 2.2. $\pi_1(B\mathcal{Q}\mathcal{A}) = \mathrm{K}_0(\mathcal{A})$

Proof. Let T denote the family of all maps $0 \hookrightarrow X$ in $\mathcal{Q}\mathcal{A}$. Note that T is a maximal tree, and so by basic algebraic topology tools, we know $\pi_1(B\mathcal{Q}\mathcal{A})$ is generated by the maps in $\mathcal{Q}\mathcal{A}$, modulo the relations $[0 \hookrightarrow X] = 1$ and $[f][g] = [fg]$.

One can observe without much work that, for any map in $\mathcal{Q}\mathcal{A}$, the following holds:

$$[X \twoheadleftarrow Y' \hookrightarrow Y] = [X \twoheadleftarrow Y'] = [0 \twoheadleftarrow Y'][0 \twoheadleftarrow X]^{-1},$$

so we see that the class of a map is given by its "epi part", and the maps $[0 \twoheadleftarrow X]$ generate the group.

Now, given an exact sequence

$$0 \to X \hookrightarrow Y \twoheadrightarrow X \to 0,$$

one can easily observe that

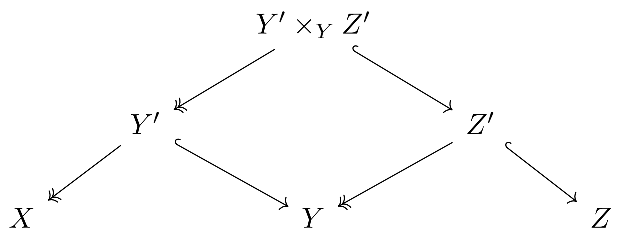
$$0 \twoheadleftarrow X \hookrightarrow Y = 0 \hookrightarrow Z \twoheadleftarrow Y$$

and so

$$[0 \twoheadleftarrow X] = [Z \twoheadleftarrow Y] = [0 \twoheadleftarrow Y][0 \twoheadleftarrow Z]^{-1}$$

which yields the desired relation. Moreover, we see that the class of an epi is uniquely determined by the class of its kernel.

Finally, any relation $[f][g] = [fg]$ can be obtained through the additivity relation we get from exact sequences. To see this, note that requiring the above relation amounts to requiring, in the composition diagram



that $[Y' \twoheadleftarrow Y' \times_Y Z'] = [Y \twoheadleftarrow Z]$, but since both epis have the same kernel, the two classes must be the same. □

Aside from checking a basic requirement, this proof gives us some intuition as to why $\Omega B\mathcal{Q}\mathcal{A}$ is a reasonable space to define K-theory. As we mentioned before, recall that an alternate construction of K_0 would be the following: take the monoid generated by iso classes $[X]$, with addition given by direct sum, $[X] + [Y] = [X \oplus Y]$. Then, take the group completion of this monoid, adding the formal inverses that were not already present in the monoid. Finally, quotient out by the extra relations given by the exact sequences.

Looking at our space, we see that ΩB acts as a "homotopical group completion" in some sense. Then, any category that occupies the place of $\mathcal{Q}\mathcal{A}$ should be built so that it records all the relevant information in $\mathcal{A}$, and so that, in the group completion, the relations are exactly the ones given by the short exact sequences.

The category $\mathcal{Q}\mathcal{A}$ is engineered for this very purpose: the maps in $\mathcal{Q}\mathcal{A}$ encode the information present in the short exact sequences, and moreover, come with a canonical epi-mono factorization that is built into the way composition is defined in the category; these are the key facts used in the proof above, and something that would not be true if we simply considered the category $\mathcal{A}$, even if it was abelian.

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