An introduction to locally finitely presentable categories

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A document born out of my attempt to understand the notion of locally finitely presentable category, and my annoyance at constantly finding different definitions.

When working with sets, the structure of a set X is completely determined by its elements. Since we can think about elements of X as maps $\{*\} \to X$, we get that mapping out of $\{*\}$ allows us to completely understand the objects of **Set**.

However, even when dealing with categories that have a terminal object, this is not usually the case. Consider for example the category of non-directed graphs (where loops are allowed, but no multiple edges). In this case, the terminal object T is the graph with only one vertex and one loop, so if X is a discrete graph, we won't even have any maps $T \to X$.

To get rid of this issue, we might try to delete the loop and instead consider the graph $\bullet$, but maps $\bullet \to X$ don't yield enough information, since they only involve the vertices of X and there can be many graphs with the same set of vertices. A similar thing happens if we only look at maps $(\bullet - \bullet) \to X$, since these don't see discrete vertices. But if we keep in mind both of these graphs, along with the incidence relations between them (which say that the point $\bullet$ lies in the edge $\bullet - \bullet$ in two ways), then we can determine the graph X.

In this case, instead of a "test object", we have a "test category", namely,

$$\bullet \rightrightarrows (\bullet - \bullet)$$

We would like to do this with other categories, that is, find a suitable subcategory $\mathcal{G} \subset \mathcal{C}$ such that the objects of $\mathcal{G}$ are "nice" and that maps out of them (together with the incidence maps in $\mathcal{G}$) determine the objects of $\mathcal{C}$.

Building up to the definition

What should we take "nice" to mean? Well, one possible criterion would be to ask for the objects of $\mathcal{G}$ to be finite in some sense, since we're usually better at understanding finite things. Let's introduce some concepts that will allow us to see what the natural condition for finiteness is in some concrete categories.

Definition 1. A *filtered category* is a category $\mathcal{C}$ in which every finite diagram has a cocone. This is the categorification of directed sets: in addition to having an upper bound for pairs of objects (meaning, for every pair of objects X_1, X_2, another object Y with maps $X_i \to Y$), there must also be an upper bound for pairs of parallel morphisms (i.e., given $X \rightrightarrows Y$, a map $Y \to Z$ such that the two compositions $X \rightrightarrows Y \to Z$ are equal).

Definition 2. A *filtered colimit* is a colimit of a functor $D : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a filtered category.

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Example 3. When $\mathcal{C} = \mathbf{Set}$, a finite object is just a finite set. Is there a "categorical" way to characterize these objects?

Proposition 4. *A set X is finite if and only if the functor* $\mathrm{Hom}(X, -) : \mathbf{Set} \to \mathbf{Set}$ *preserves filtered colimits.*

Proof. Every set is the filtered colimit of its finite subsets $X_i \subset X$ and inclusions between them. Thus, the condition that $\mathrm{Hom}(X, -)$ preserves filtered colimits implies that the canonical comparison map

$$\operatorname{colim} \mathrm{Hom}(X, X_i) \to \mathrm{Hom}(X, X)$$

is a bijection, and some element in the colimit represented by $f : X \to X_i$ gets mapped to 1_X, i.e., $i \circ f = 1_X$ for some inclusion $i : X_i \hookrightarrow X$. This implies i is a bijection, so X is a finite set.

For the converse, the functor $\mathrm{Hom}(\{*\}, -)$ is the identity functor, so it preserves colimits. Furthermore, any set X with n elements can be written as $X \simeq \coprod_n \{*\}$, so

$$\mathrm{Hom}(X, -) \simeq \mathrm{Hom}(\coprod_n \{*\}, -) \simeq \lim_n \mathrm{Hom}(\{*\}, -)$$

and then $\mathrm{Hom}(X, -)$ preserves filtered colimits, since limits and filtered colimits commute in $\mathbf{Set}$. □

Example 5. When $\mathcal{C} = \mathbf{Vect}_{\Bbbk}$, a finite object is a finite-dimensional vector space.

Proposition 6. *A vector space V is finite-dimensional if and only if the functor* $\mathrm{Hom}(V, -) : \mathbf{Vect}_{\Bbbk} \to \mathbf{Set}$ *preserves filtered colimits.*

Proof. Every vector space V is the filtered colimit of its finite-dimensional subspaces $V_i \subseteq V$, so, much as in the previous example, if $\mathrm{Hom}(V, -) : \mathbf{Vect}_{\Bbbk} \to \mathbf{Set}$ preserves filtered colimits, then V must be finite-dimensional.

In the converse direction, we know that for any finite dimensional V we have an adjunction $(- \otimes_k V) \dashv (- \otimes V^*)$, which in turn yields an adjunction $\mathrm{Hom}(V, -) \dashv \mathrm{Hom}(V^*, -)$. Then, as a left adjoint, $\mathrm{Hom}(V, -)$ preserves all colimits (and in particular filtered colimits). □

Example 7. Let R be a ring. We could argue that there are two natural notions for what a "finite" R-module should be: a finitely generated module, or a finitely presented one. The second one is stronger (since it requires finitely many generators with finitely many relations between them), and one can show that a module M is finitely presented if and only if the functor $\mathrm{Hom}(M, -) : R\text{-Mod} \to \mathbf{Set}$ preserves filtered colimits.

Taking the cue from the previous examples, we define the following notion.

Definition 8. An object X in $\mathcal{C}$ is *finitely presented* (or compact) if the functor $\mathrm{Hom}(X, -) : \mathcal{C} \to \mathbf{Set}$ preserves filtered colimits. This is the same as saying that if $D : \mathcal{I} \to \mathcal{C}$ is a diagram where $\mathcal{I}$ is a filtered category, then any morphism $X \to \operatorname{colim} D_i$ factors (essentially uniquely) through some $X \to D_i$.

For a category $\mathcal{C}$, let $\mathcal{C}_{fp}$ denote the full subcategory of finitely presented objects.

Example 9. Finitely presented graphs are those with finitely many vertices and edges.

Example 10. The categorical notion of finite presentability for groups coincides with the algebraic one; that is, for a group G, $\mathrm{Hom}(G, -)$ preserves filtered colimits if and only if G has finitely many generators with finitely many relations among them.

In fact, this case, together with examples 5 and 7, are particular cases of a larger fact: in the category of algebras for any algebraic theory, the categorical and algebraic notions of finite presentability agree ([AR94, Thm. 3.12]).

The proof of proposition 4 can be generalized to show the following.

Proposition 11. *A finite colimit of finitely presentable objects is again finitely presentable.*

The definition(s)

Now that we know what our "nice" objects look like, we need to decide which categories we want to consider. Let's remember our initial goal: we want a category $\mathcal{C}$ such that the objects of $\mathcal{C}$ are determined by maps from objects in $\mathcal{C}_{fp}$, and incidence maps within $\mathcal{C}_{fp}$. Given that we need certain colimits to exist in order to talk about finitely presented objects in $\mathcal{C}$, it makes sense to require $\mathcal{C}$ to be cocomplete. Also, since we will be probing with objects in $\mathcal{C}_{fp}$, it seems reasonable to limit the size of $\mathcal{C}_{fp}$: we will ask for it to be skeletally small. That leads us to the following.

Definition 12. A locally small category $\mathcal{C}$ is *locally finitely presentable* if it has all small colimits, $\mathcal{C}_{fp}$ is skeletally small, and the functor $F : \mathcal{C} \to [\mathcal{C}_{fp}, \mathbf{Set}]$ defined by $F(X)(A) = \mathrm{Hom}(A, X)$ (a restriction of the Yoneda embedding) is faithful and conservative.

Recall that a functor is called conservative if it reflects isomorphisms; this implies that looking at $\mathcal{C}(-, X) : \mathcal{C}_{fp} \to \mathbf{Set}$ actually determines the object X up to isomorphism. In fact, if the category $\mathcal{C}$ has equalizers, we can show that this is all we really need to ask of the functor F.

Proposition 13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a conservative functor. If $\mathcal{C}$ has equalizers and F preserves them, then F is faithful.*

Proof. Let $f, g : X \rightrightarrows Y$ be a pair of parallel arrows, and consider the equalizer

$$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Since F preserves equalizers, we know that

$$FE \xrightarrow{Fe} FX \underset{Fg}{\overset{Ff}{\rightrightarrows}} FY$$

will also be an equalizer. Now suppose $Ff = Fg$; that's the same as saying that the equalizer of Ff and Fg is $1_{FX} : FX \to FX$, and so there exists an isomorphism $\varphi : FE \to FX$ such that $Fe = 1_{FX}\varphi$. But then Fe is an isomorphism, and since F is conservative, that means e is an isomorphism. Thus, from $fe = ge$, we get that $f = fee^{-1} = gee^{-1} = g$. $\square$

In our case, note that the functor $F : \mathcal{C} \to [\mathcal{C}_{fp}, \mathbf{Set}]$ preserves limits (and in particular, equalizers), since limits in $[\mathcal{C}_{fp}, \mathbf{Set}]$ are computed pointwise and each $\mathcal{C}(A, -)$ preserves limits.

This seems to work very nicely, but what if we go a different way? We have a subcategory $\mathcal{C}_{fp}$ of finitely presented objects, the "suitably finite" objects in $\mathcal{C}$. What if we ask that all objects of $\mathcal{C}$ are built from these in a reasonable manner?

Definition 14. A locally small category $\mathcal{C}$ is *locally finitely presentable* if it has all small colimits, $\mathcal{C}_{fp}$ is skeletally small, and any object in $\mathcal{C}$ is the filtered colimit of a diagram in $\mathcal{C}_{fp}$.

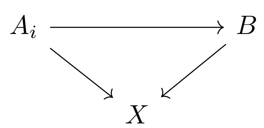
In other words, we should be able to obtain any object by adequately "gluing" together finitely presented objects, taking into account the incidence relations between them. Our algebraic intuition indicates that this is a reasonable thing to attempt, since in algebra we decompose objects into colimits of smaller, simpler objects all the time.

Indeed, if X is the colimit of a diagram in $\mathcal{C}_{fp}$, we will have certain maps from some objects of $\mathcal{C}_{fp}$ to X (the legs of the colimit cocone), but how does this relate to our initial goal of having X be *determined* by all maps from objects in $\mathcal{C}_{fp}$ to X?

Proposition 15. *An object X in $\mathcal{C}$ can be written as a filtered colimit of objects in $\mathcal{C}_{fp}$ if and only if X is the colimit of the diagram $U_X : C_{fp} \downarrow X \to \mathcal{C}$, where U_X is the forgetful functor.*

Proof. If we assume that X is the colimit of the diagram $U_X : C_{fp} \downarrow X \to \mathcal{C}$, then all we need to show is that the category $C_{fp} \downarrow X$ is filtered.

There's an upper bound for every pair of objects: given two objects $A_1 \to X, A_2 \to X$, we want another object $B \to X$ with maps $(A_i \to X) \to (B \to X)$ in $C_{fp} \downarrow X$, i.e. maps $A_i \to B$ in $\mathcal{C}$ such that the diagram commutes

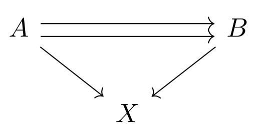


It's easy to show that $B = A_1 \coprod A_2$ is finitely presented, and the universal property of the coproduct gives us all the maps we need.

There's an upper bound for every parallel pair of maps: consider a diagram

$$(A \to X) \rightrightarrows (B \to X)$$

in $C_{fp} \downarrow X$, or equivalently, the commutative diagram



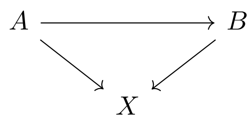
in $\mathcal{C}$. We want an object $(C \to X)$ in $C_{fp} \downarrow X$ and a map $(B \to X) \to (C \to X)$ such that the two compositions

$$(A \to X) \rightrightarrows (B \to X) \to (C \to X)$$

are equal, or equivalently, such that the compositions $A \rightrightarrows B \to C$ are the same in $\mathcal{C}$. Due to the previous commutative triangle, we achieve this by letting $(C \to X) = (B \to X)$.

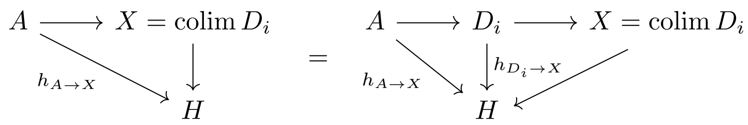
Now, suppose that X is the colimit of some diagram $D : \mathcal{I} \to \mathcal{C}_{fp} \hookrightarrow \mathcal{C}$ where $\mathcal{I}$ is filtered. The maps $D_i \to \operatorname{colim} D_i = X$ are some of the objects considered in $C_{fp} \downarrow X$; we need to show that they somehow determine the colimit of $U_X : C_{fp} \downarrow X \to \mathcal{C}$.

First of all, let's show that $\operatorname{colim} D_i$ forms a cocone. For each object $A \to X = \operatorname{colim} D_i$ in $\mathcal{C}_{fp} \downarrow X$, there exists a map (itself) $U_X(A \to \operatorname{colim} D_i) = A \to \operatorname{colim} D_i$. Clearly, since a map $(A \to X) \to (B \to X)$ in $C_{fp} \downarrow X$ is given by a commutative diagram



all the legs of the cocone commute.

Given another cocone nadir H with maps $\{U_X(A \to X) = A \xrightarrow{h_{A \to X}} H\}_{A \to X}$, we can restrict it to the cocone $\{D_i \xrightarrow{h_{D_i \to X}} H\}$ to get a map $X = \operatorname{colim} D_i \to H$ that commutes with the cocone legs involving the D_i's. Given any $A \to X = \operatorname{colim} D_i$, the fact that $\operatorname{colim} D_i$ is a filtered colimit implies that map must factor through some $D_i \to \operatorname{colim} D_i$. Thus, we have the diagrams



where, if we focus on the diagram on the right, the right triangle commutes from the construction of the map $\operatorname{colim} D_i \to H$, and the left triangle on the left commutes because H is a cocone. That means the diagram on the left commutes, which proves that every other cocone factors through $\operatorname{colim} D_i$, and therefore $\operatorname{colim} D_i$ is the colimit of $U_X : \mathcal{C}_{fp} \downarrow X \to \mathcal{C}$. □

This shows we have two reasonable candidates for our definition of locally finitely presentable category. Let's see how they relate to each other.

Theorem 16. *Let $\mathcal{C}$ be a cocomplete, locally small category. Then every object X of $\mathcal{C}$ is the colimit of the diagram $U_X : C_{fp} \downarrow X \to \mathcal{C}$ if and only if the functor $F : \mathcal{C} \to [\mathcal{C}_{fp}, \mathbf{Set}]$ defined by $F(X)(A) = \operatorname{Hom}(A, X)$ is fully faithful.*

Proof. F being fully faithful means there exists a bijection

$$\mathcal{C}(X, Y) \simeq \operatorname{Nat}(\mathcal{C}(-, X), \mathcal{C}(-, Y))$$

for every X, Y in $\mathcal{C}$, where the hom-sets on the right are restricted to $\mathcal{C}_{fp}$. Now, there is a bijective correspondence

$$\operatorname{Nat}(U_X, \Delta_Y) \simeq \operatorname{Nat}(\mathcal{C}(-, X), \mathcal{C}(-, Y))$$

since an element on the left is a natural transformation $\tau : U_X \to \Delta_Y$, so, for each $A \to X$ in $\mathcal{C}_{fp} \downarrow X$, we have a map $\tau_{A \to X} : A \to Y$, and an element on the right is a natural transformation $\hat{\tau} : \mathcal{C}(-, X) \to \mathcal{C}(-, Y)$, meaning, for each for A in $\mathcal{C}_{fp}$, it takes a map $A \to X$ to a map $\hat{\tau}_A(A \to X) : A \to Y$.

Finally, given an object X in $\mathcal{C}$, there exists a bijection

$$\mathcal{C}(X, Y) \simeq \mathrm{Nat}(U_X, \Delta_Y)$$

for every Y in $\mathcal{C}$ if and only if X represents the functor $\mathrm{Nat}(U_X, \Delta_-)$, which is precisely the definition of X being the colimit of the diagram U_X. □

Theorem 17. *Definitions 12 and 14 of a locally finitely presentable category agree. That is, if $\mathcal{C}$ is a cocomplete category and $\mathcal{C}_{fp}$ is skeletally small, then the functor $F : \mathcal{C} \to [\mathcal{C}_{fp}, \mathbf{Set}]$ defined by $F(X)(A) = \mathrm{Hom}(A, X)$ is faithful and conservative if and only if any object in $\mathcal{C}$ is the filtered colimit of a diagram in $\mathcal{C}_{fp}$.*

Proof. If every object in $\mathcal{C}$ is the filtered colimit of a diagram in $\mathcal{C}_{fp}$, then proposition 15, along with theorem 16 and the fact that fully faithful functors are conservative, gives us our result.

For the converse, recall that X together with all maps $\{A \to X\}$ form a cocone for U_X (as shown in the proof of prop. 15), so we have a map $\varphi : \operatorname{colim} U_X \to X$. We will show that φ is an isomorphism.

To prove it's a monomorphism, suppose $g, h : Y \to \operatorname{colim} U_X$ are two maps such that $\varphi g = \varphi h$. The category $\mathcal{C}_{fp} \downarrow X$ is filtered, so g and h can be factored as

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & \operatorname{colim} U_X \\ & {\scriptstyle g'}\searrow \quad \nearrow{\scriptstyle f_{A\to X}} & \\ & U_X(A \to X) & \end{array} \quad , \quad \begin{array}{ccc} Y & \xrightarrow{\quad h \quad} & \operatorname{colim} U_X \\ & {\scriptstyle h'}\searrow \quad \nearrow{\scriptstyle f_{B\to X}} & \\ & U_X(B \to X) & \end{array}$$

Again, since $\mathcal{C}_{fp} \downarrow X$ is filtered, there exists an object $C \xrightarrow{c} X$ and maps $A \to C$, $B \to C$ such that the diagrams

$$\begin{array}{ccc} A & \longrightarrow & C \\ & \searrow \quad \swarrow & \\ & X & \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & C \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commute. Thus, we can factor g and h through the same object $C \to X$, getting

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & \operatorname{colim} U_X \\ & {\scriptstyle \bar{g}}\searrow \quad \nearrow{\scriptstyle f_{C\to X}} & \\ & C & \end{array} \quad , \quad \begin{array}{ccc} Y & \xrightarrow{\quad h \quad} & \operatorname{colim} U_X \\ & {\scriptstyle \bar{h}}\searrow \quad \nearrow{\scriptstyle f_{C\to X}} & \\ & C & \end{array}$$

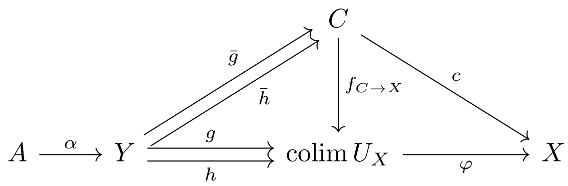
Given that the functor F is faithful, showing that $g = h$ is equivalent to proving that the natural transformations $Fg, Fh : \mathcal{C}(-, Y) \Rightarrow \mathcal{C}(-, \operatorname{colim} U_X)$ are equal; that is, for any finitely presented A and $\alpha : A \to Y$, we have $g\alpha = h\alpha$.

Consider the coequalizer

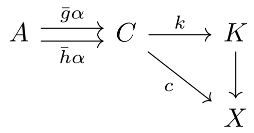
$$A \underset{\bar{h}\alpha}{\overset{\bar{g}\alpha}{\rightrightarrows}} C \xrightarrow{\ k\ } K$$

Both A and C are in $\mathcal{C}_{fp}$, so by prop. 11, so is K.

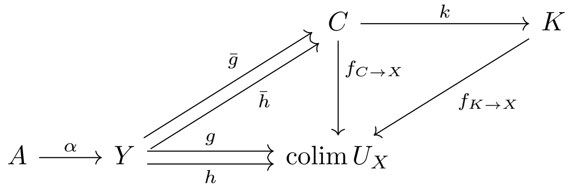
Now, if we record all our information in the diagram



we see that $c\bar{g}\alpha = c\bar{h}\alpha$, so there exists a map $K \to X$ that makes the following diagram commute



Since K is finitely presented, the map $K \to X$ is an object of $\mathcal{C}_{fp} \downarrow X$, and the triangle in the previous diagram exhibits k as a map from $(C \to X)$ to $(K \to X)$. Therefore, we have the following situation



and if we recall that K is the coequalizer of $\bar{g}\alpha$ and $\bar{h}\alpha$, we see that $g\alpha = h\alpha$ as we wanted.

Note that in order to show that φ is an isomorphism, it suffices to prove that $F\varphi = \varphi \circ - : \mathcal{C}(-, \operatorname{colim} U_X) \Rightarrow \mathcal{C}(-, X)$ is an epimorphism, i.e. that $\varphi \circ - : \mathcal{C}(A, \operatorname{colim} U_X) \to \mathcal{C}(A, X)$ is a surjection for every finitely presentable A. Indeed, we already know that φ is a monomorphism, so $F\varphi$ is a mono as well (since F clearly preserves monomorphisms). Therefore, showing $F\varphi$ is an epi would imply that it's an isomorphism, hence reaching our conclusion because F is conservative.

Finally, $\varphi \circ - : \mathcal{C}(A, \operatorname{colim} U_X) \to \mathcal{C}(A, X)$ is a surjection since we can exhibit a right inverse, taking a map $A \to X$ to its corresponding colimit leg $A \xrightarrow{f_{A \to X}} \operatorname{colim} U_X$. □

Polishing the definition

Sometimes, considering the whole subcategory $\mathcal{C}_{fp}$ might be an overkill. Going back to the case of graphs, example 9 tells us that the finitely presented graphs are those with finitely many vertices and edges. However, in the introduction we mentioned that we really just need to consider the subcategory

$$\bullet \rightrightarrows \bullet - \bullet$$

if we want to determine an object. This suggests the following reformulation.

Definition 18. A locally small category $\mathcal{C}$ is *locally finitely presentable* if it has all small colimits, and there exists a set S of objects of $\mathcal{C}_{fp}$ such that any object in $\mathcal{C}$ is the filtered colimit of a diagram with objects in S.

Clearly, definition 14 implies definition 18, by taking S to be a set of representatives of the isomorphism classes of finitely presented objects. To see that they are equivalent, we need to show that definition 18 implies that $\mathcal{C}_{fp}$ is skeletally small, which is not readily apparent. We proceed as suggested in [AR94, Excercise 1.d(2)].

First of all, we show that $\mathcal{C}$ is well-powered (i.e. every object has a set's worth of subobjects, up to isomorphism). Let $\mathcal{S}$ denote the small full subcategory of $\mathcal{C}$ whose set of objects is S. Since the restriction of the Yoneda embedding $F_S : \mathcal{C} \to [\mathcal{S}, \mathbf{Set}]$ preserves monomorphisms, it suffices to show that $[\mathcal{S}, \mathbf{Set}]$ is well-powered.

Let $F : \mathcal{S} \to \mathbf{Set}$ be a functor, and let $G : \mathcal{S} \to \mathbf{Set}$ be a subobject of F, meaning, there exists a monomorpism $\tau : G \Rightarrow F$. For each $X \in \mathcal{S}$ there is an injection $\tau_X : GX \to FX$, so GX is a subobject of FX. Now, FX is a set, so clearly it has a set's worth of subobjects (up to isomorphism); let $\{Z_j^X \xrightarrow{f_j^X} FX\}_{j \in I_X}$ be a set of representatives. That means there exists some $i \in I_X$ and an isomorphism $\varphi_X : GX \to Z_i^X$ such that $\tau_X = \varphi_X f_i^X$.

We want to define a functor $G' : \mathcal{S} \to \mathbf{Set}$ such that G' is a subobject of F and $G \simeq G'$. Since we already have suitable maps $GX \xrightarrow{\varphi_X} Z_i^X \xrightarrow{f_i^X} FX$, it makes sense to define $G'X = Z_i^X$. For a map $g : X \to Y$, we can see from the commutative diagram

$$\begin{array}{ccccc} GX & \xrightarrow{\varphi_X} & Z_i^X = G'X & \xrightarrow{f_i^X} & FX \\ \downarrow{\scriptstyle Gg} & & & & \downarrow{\scriptstyle Fg} \\ GY & \xrightarrow{\varphi_Y} & Z_i^Y = G'Y & \xrightarrow{f_i^Y} & FY \end{array}$$

that defining $G'g = \varphi_Y Gg \varphi_X^{-1}$ makes $\varphi_- : G \Rightarrow G'$ and $f_i^- : G' \Rightarrow F$ into natural transformations, with $\tau = \varphi_- \circ f_i^-$.

Finally, note that we only have a set's worth of subobjects of the form $f_i^- : G' \Rightarrow F$, since each f_i^- is an element of the set $\prod_{X \in S} I_X$. This shows that $\mathcal{C}$ is well-powered.

Now, let A be a finitely presented object, and let $D : \mathcal{I} \to \mathcal{S} \hookrightarrow \mathcal{C}$ be a diagram such that $A = \operatorname{colim} D_i$, with I filtered. Then, as any map to a filtered colimit, $1_A : A \to A$ factors through some D_i as

$$\begin{array}{ccc} A & \xrightarrow{\quad 1_A \quad} & A = \operatorname{colim} D_i \\ & \searrow \quad \nearrow & \\ & D_i & \end{array}$$

which means A is a subobject of D_i. We have proved that any finitely presented object is a subobject of an object in S, so the fact that S is a set, together with $\mathcal{C}$ being well-powered, implies that $\mathcal{C}_{fp}$ must be skeletally small.

Similarly, we could alter definition 12 in the following manner.

Definition 19. A locally small category $\mathcal{C}$ is *locally finitely presentable* if it has all small colimits, and there exists a small subcategory $\mathcal{S} \subset \mathcal{C}_{fp}$ such that the functor $F : \mathcal{C} \to [\mathcal{S}, \mathbf{Set}]$ defined by $F(X)(A) = \mathrm{Hom}(A, X)$ is faithful and conservative.

To show that definitions 18 and 19 are equivalent we can consider $\bar{\mathcal{S}}$, the closure of $\mathcal{S}$ under finite colimits (noting that it will be skeletally small) and proceed as in theorem 17, substituting $\mathcal{C}_{fp}$ by $\bar{\mathcal{S}}$.

Some examples, and yet another characterization

Example 20. The category **Set** is l.f.p., with $\mathbf{Set}_{fp} = \{\text{finite sets}\}$.

Example 21. The category of graphs is l.f.p., for example, with $S = \{\bullet, \bullet - \bullet\}$.

Example 22. The category of small categories, **Cat**, is l.f.p., for example with $S = \{\bullet \to \bullet\}$.

Example 23. The category **Top** is **not** l.f.p. (I have to think about this).

Example 24. The category of algebras for any algebraic theory is l.f.p., and we can take $S = \{\text{free algebra on one generator}\}$.

Actually, a generalization of this last example gives us the only class of examples that we can find.

Theorem 25. *[Kel82, Thm. 9.8] A category $\mathcal{C}$ is l.f.p. if and only if it is the category of algebras for a finitary essentially algebraic theory (which is then $\mathcal{T} = \mathcal{C}_{fp}^{op}$).*

Definition 26. A finitary essentially algebraic theory $\mathcal{T}$ is a small category that has all finite limits. A model (or algebra) of a theory $\mathcal{T}$ is a left exact functor $M : \mathcal{T} \to \mathcal{C}$.

Remark 27. Recall that left exact functors are precisely those that preserve small limits.

These generalize finitary algebraic theories (aka Lawvere theories) where instead of finite limits we only consider finite products.

Morally (from the nLab), "a mathematical structure is essentially algebraic if its definition involves partially defined operations satisfying equational laws, where the domain of any given operation is a subset where various other operations happen to be equal. An actual algebraic theory is one where all operations are total functions."

Some examples, aside from Lawvere theories themselves, are categories, where composition is only defined for certain pairs of morphisms.

Summing up

Recapitulating, a locally small, cocomplete category is locally finitely presentable if any of the following equivalent conditions are satisfied:

1. $\mathcal{C}_{fp}$ is skeletally small, and any object in $\mathcal{C}$ is the filtered colimit of a diagram in $\mathcal{C}_{fp}$.

2. There exists a set S of objects of $\mathcal{C}_{fp}$ such that any object in $\mathcal{C}$ is the filtered colimit of a diagram with objects in S.

3. $\mathcal{C}_{fp}$ is skeletally small, and the functor $F : \mathcal{C} \to [\mathcal{C}_{fp}, \mathbf{Set}]$ defined by $F(X)(A) = \mathrm{Hom}(A, X)$ is fully faithful.

4. $\mathcal{C}_{fp}$ is skeletally small, and the functor $F : \mathcal{C} \to [\mathcal{C}_{fp}, \mathbf{Set}]$ defined by $F(X)(A) = \mathrm{Hom}(A, X)$ is faithful and conservative.

5. There exists a small subcategory $\mathcal{S} \subset \mathcal{C}_{fp}$ such that the functor $F : \mathcal{C} \to [\mathcal{S}, \mathbf{Set}]$ defined by $F(X)(A) = \mathrm{Hom}(A, X)$ is faithful and conservative.

6. $\mathcal{C}$ is the category of algebras for a finitary essentially algebraic theory.

If, in addition, $\mathcal{C}$ has equalizers, the adjective "faithful" can be dropped from conditions 4 and 5.

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