MATH 4530 – Topology. Practice Problems For Final Part II

(1) (One point compactification) Assume that X is a non-compact connected Hausdorff space in which every point has a compact neighborhood. Define $X' := X \sqcup \{\infty\}$ as a set. You may use the fact that the intersection of a family of compact sets in a Hausdorff space is compact and the fact that the union of a finite collection of compact sets is compact.
  (a) Define a topology on X' as follows: a subset $U \subset X'$ is open if (i) it is an open subset of X if $U \subset X$, and (ii) $X' - U$ is a compact subset in X if it is not a subset of X. Prove that this actually defines a topology on X'.
  (b) Show that X is a subspace of X'.
  (c) Show that X' is compact.
  (d) Show that X' is connected.
  (e) Show that if X is $\mathbb{R}^2$ with the usual topology, then X' is homeomorphic to the 2-sphere S^2.

(2) (a) Explain why (and how) a continuous map $f : X \to Y$ with $f(x) = y$ induces a group homomorphism $\pi_1(X, x) \to \pi_1(Y, y)$.
  (b) Use the fact that $\pi_1(S^1) \cong \mathbb{Z}$ to prove Brouwer's Fixed Point Theorem: for every continuous map $f : D^2 \to D^2$, there is $a \in D^2$ such that $f(a) = a$.

(3) Let X be any topological space, Y a Hausdorff space, and $f : X \to Y$ a continuous map. The graph of f is defined as the subspace

$$G_f := \{(x, f(x)) \in X \times Y \mid x \in X\}.$$

  (a) Show that G_f is a closed subspace.
  (b) Find a counter example to part (a) in the case when Y is not Hausdorff.
  (c) If $f : X \to Y$ is a map and G_f is closed, then f must be continuous?

(4) Let X be a topological space, and A and B compact subspaces.
  (a) Show that $A \cup B$ is compact.
  (b) Show that if X is Hausdorff, then $A \cap B$ is compact.
  (c) Give a counterexample to part (b) in the case when X is not Hausdorff.

(5) (a) Let X be a Hausdorff space. Show that any connected subset $A \subset X$ contains one or infinitely many elements.
  (b) Let A be a countable subset of $\mathbb{R}^2$. Prove that $\mathbb{R}^2 - A$ is path-connected.

(6) Determine whether or not there is a retraction from X to A for the following spaces. If there is a retraction, describe it explicitly, using pictures if you like.
  (a) X is $S^1 \times D^2$ and A is $S^1 \times S^1$.
  (b) X is $S^1 \times S^1$ and $A = \{(x, x) \in X \mid x \in S^1\}$.

(7) Prove that a surjective map from a compact space to a Hausdorff space is a quotient map.

(8) Prove that $S^1 := \{e^{2\pi i\theta}, \theta \in \mathbb{R}\} \subset \mathbb{C}$ is homeomorphic to the quotient space obtained from $[0, 1]$ by identifying 0 and 1.

References

[M] Munkres, Topology.
[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf
[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html