MATH 4530 – Topology. Prelim I Solutions

In class 75min: 2:55-4:10 Thu 9/30.

Problem 1: Consider the following topological spaces:

(1) $\mathbb{Z}$ as a subspace of $\mathbb{R}$ with the finite complement topology
(2) $[0,\pi]$ as a subspace of $\mathbb{R}$.
(3) $[0,\pi]-\{1\}$ as a subspace of $\mathbb{R}$.
(4) The unit n-sphere $S^n := \{(x_1,\cdots,x_{n+1})\in\mathbb{R}^n \mid x_1^2+\cdots+x_{n+1}^2 = 1\}$ as a subspace of $\mathbb{R}^{n+1}$.
(5) The unit n-disk $D^n := \{(x_1,\cdots,x_{n+1})\in\mathbb{R}^n \mid x_1^2+\cdots+x_{n+1}^2 \leq 1\}$ as a subspace of $\mathbb{R}^{n+1}$.
(6) $S^1\times S^1$ with the product topology, called the ***torus***.
(7) $\mathbb{RP}^2$ defined as a quotient space of $\mathbb{R}^3-\{(0,0,0)\}$ by the identification $(x,y,z)\sim(\lambda x,\lambda y,\lambda z)$ for all $\lambda\in\mathbb{R}-\{0\}$.
(8) The quotient space of $\mathbb{R}^2$ by the identification $(x,y)\sim(\lambda x,\lambda y)$ for all $\lambda\in\mathbb{R}_{>0}$.
(9) $\mathrm{O}(2,\mathbb{R})$ the set of all 2×2 matrices M with real values such that $MM^t = I_2$, as a subspace of $\mathbb{R}^4$.
(10) The quotient space of $\mathbb{R}^2$ by the identification $(x,y)\sim(x+n,y+n)$ for all $n,m\in\mathbb{Z}$.
(11) The closure of $\{(x^2,\sin(1/x))\in\mathbb{R}^2 \mid x\in\mathbb{R}_{>0}\}$ in $\mathbb{R}^2$.

Q1-1: Find *all* compact spaces. (11pts)

Solution:

(1) Compact: *Any infinite set with finite complement topology is compact.* The proof is as follows. Let X be an infinite set with the f.c. topology. Let $\{U_\alpha\}$ be a covering of X. Then $X-U_\alpha$ is a finite set, say $\{x_1,\cdots,x_n\}$. Let U_{α_i} be one of the open sets that contains x_i. Then $U_\alpha\cup U_{\alpha_1}\cup\cdots\cup U_{\alpha_n} = X$.
(2) Compact: This is the most basic key fact of compactness.
(3) Not compact: Let $U_n := [0,1)\cup(1+1/n,\pi]$ for $n\in\mathbb{Z}_{>0}$. Then $\{U_n\}$ is an open covering. If there is a finite subcovering, say, $\{U_{n_i}\}_{i=1,\cdots,m}$. Let $a := \max\{n_1,\cdots,n_m\}$. Then $\cup_{i=1}^m U_{n_i}$ cannot contain $1+1/(a+1)$. So it can't be a covering.
(4) Compact: Any closed, bounded subset in $\mathbb{R}^n$ is compact.
(5) Compact: Any closed, bounded subset in $\mathbb{R}^n$ is compact.
(6) Compact: S^1 is compact from (4). A finite product of compact spaces is compact.
(7) Compact: It is actually a quotient of $S^2\subset\mathbb{R}^3-\{(0,0,0)\}$. Since S^3 is compact from (4), $\mathbb{RP}^2$ is the image of the quotient map of a compact space. Since the image of a compact space under a continuous map is compact, it is compact.
(8) Compact: The quotient map factors through a surjective map $S^3\cup\{(0,0,0)\}\to\mathbb{R}^2/\sim$. This map is not quotient map but it is still continuous (the restriction of a continuous map on a subspace is continuous). Since $S^3\cup\{(0,0,0)\}$ is bounded and closed, it is compact. Thus $\mathbb{R}^2/\sim$ is the image of a compact space, so it is compact.
(9) Compact: The row vectors $\vec{v}_1,\vec{v}_2$ of orthogonal matrices are unit vectors. As a vector in $\mathbb{R}^4$, $(\vec{v}_1,\vec{v}_2)$ has length 2. So $\mathrm{O}(2,\mathbb{R})$ is bounded, actually is contained in S^3. Consider the continuous map $M\mapsto M\cdot M^t$. Then $\mathrm{O}(2,\mathbb{R})$ is the pre-image of I a point in $\mathbb{R}^4$, thus it is closed. Closed and bounded subsets in $\mathbb{R}^4$ are compact.
(10) Compact: The quotient map restricted to the compact supspace $[0,1]\times[0,1]$ is surjective continuous. Thus the quotient is compact.
(11) Not compact: It is not compact since it is not bounded.

Q1-2: Find *all* connected spaces. (11pts)

Solution:

(1) Connected: any infinite set with a f.c. topology is connected.
(2) Connected: it is the closure of $(0,\pi)$ which is homeomorphic to $\mathbb{R}$.
(3) Not connected: It is a disjoint union of non-empty open sets.
(4) Connected: since it is path-connected.
(5) Connected: since it is path-connected.
(6) Connected: since it is path-connected.
(7) Connected: since it is path-connected.
(8) Connected: since it is path-connected.
(9) Not connected: the determinant map $M \mapsto \det M$ sends orthogonal matrices to ± 1. The preimage of ± 1 are non-empty closed sets and so $\mathrm{O}(n,\mathbb{R})$ is a disjoint union of non-empty closed sets.
(10) Connected: since it is path-connected.
(11) Connected: since it is the closure of a connected space.

Q1-3: Find *all* Hausdorff spaces. (11pts)

Solution:

(1) Not Hausdorff: Every two open sets would intersect non-trivially in an infinite set with f.c. topology.
(2) Hausdorff: Any subspace of a Hausdorff space is Hausdoff.
(3) Hausdorff: Any subspace of a Hausdorff space is Hausdoff.
(4) Hausdorff: Any subspace of a Hausdorff space is Hausdoff.
(5) Hausdorff: Any subspace of a Hausdorff space is Hausdoff.
(6) Hausdorff: The finite product of Hausdorff space is Hausdorff.
(7) Hausdorff: We are identifying all points on a line through origin. If we have two distinct lines, there are open sets separating those two, if we take the origin out.
(8) Non-Hausdorff: Any open set around the origin intersects with any line minus origin, so we can't separate the image of the origin from other points in $\mathbb{RP}^2$.
(9) Hausdorff: Any subspace of a Hausdorff space is Hausdoff.
(10) Hausdorff: Because it is homeomorphic to (6).
(11) Hausdorff: Any subspace of a Hausdorff space is Hausdoff.

Q1-4: Find two spaces that are homeomorphic. (2pts)

Solution: (6) and (10) are homeomorphic. First $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1. The quotient space in (10) is $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, so it is homeomorphic to $S^1 \times S^1$.

Q1-5: Find *all* path connected spaces from (2) - (11). (10pts)

Solution:

(1)
(2) Path-connected: a path from x to y is given by $tx + (1 - t)y$.
(3) Not path connected: because it is not connected.
(4) Path connected: There is a continuous surjection $R^{n+1} - \{(0, \cdots, 0)\} \to S^n$ defined by $\vec{x} \mapsto \vec{x}/|\vec{x}|$. Since $R^{n+1} - \{(0, \cdots, 0)\}$ is path connected, S^n is path-connected (image of a path connected is path connected).
(5) Path-connected: the path $t\vec{x} + (1 - t)\vec{y}$ stays inside of the disk.
(6) Path-connected: the finite product of path-connected spaces are path-connected.
(7) Path-connected: similar to (4).
(8) Path-connected: similar to (4).
(9) Not path-connected: because it is not connected.
(10) Path-connected: similar to (4).
(11) Not path-connected: $\{(x, \sin(1/x)) \mid x \in \mathbb{R}_{>0}\}$ is not path connected. There is a homeomorphism $(x, \sin(1/x)) \mapsto (x^2, \sin(1/x))$, so it is not path-connected.

Problem 2 (20pts): True or false? If it is true, write the proof in complete sentences. If it is false, then give a counterexample.

(1) (5pts) If X is a Hausdorff space, then any quotient space of X is Hausdorff.

> **Solution:** False. (8) in Problem 1 is a counterexample. $\mathbb{R}^2$ is Hausdorff, but the quotient is not.

(2) (5pts) Let $f : X \to Y$ be a continuous map between topological spaces. If a sequence (x_n) converges to x in X, then $(f(x_n))$ converges to $f(x)$ in Y.

> **Solution:** True. Let U be a neighborhood of $f(x)$, then $f^{-1}(U)$ is a neighborhood of x, so there is $N > 0$ such that $x_n \in f^{-1}(U)$ for all $n > N$. Thus $f(x_n) \in U$ for all $n > N$. Thus $f(x_n)$ converges to $f(x)$.

(3) (5pts) Let $f : X \to Y$ be a continuous and injective map. Suppose that X is compact and Y is Hausdorff. If A is a closed subspace of X, then the restriction map $f|_A : A \to f(A)$ is a homeomorphism.

> **Solution:** True. Since A is a closed subspace of a compact space X, A is compact. Since $f(A)$ is a subspace of a Hausdorff space Y, $f(A)$ is Hausdorff. Since f is injective, $f|_A : A \to f(A)$ is bijective. Thus Theorem (9) implies $f|_A$ is a homeomorphism.

(4) (5pts) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases of topologies $\mathcal{T}$ and $\mathcal{T}'$. If $\mathcal{T}$ is finer than $\mathcal{T}'$, then $\mathcal{B} \supset \mathcal{B}'$.

> **Solution:** False. The standard topology $\mathcal{T}$ of $\mathbb{R}$ is finer than the finite complement topology $\mathcal{T}'$ of $\mathbb{R}$. Let $\mathcal{B}$ be the open ball basis of $\mathcal{T}$ and $\mathcal{B} = \mathcal{T}$. $\mathcal{T} \supset \mathcal{T}'$ but $\mathcal{B} \not\supset \mathcal{B}'$, i.e. an open set in the finite complement topology is open in the standard topology but it is not an open ball.

Definitions:

(1) A collection $\mathcal{T}$ of subsets of a set X is a topology if $\varnothing, X \in \mathcal{T}$ and an arbitrary union and a finite intersection of subsets in $\mathcal{T}$ are also in $\mathcal{T}$.
(2) A collection $\mathcal{B}$ of subsets of X is a basis of a topology if the members of $\mathcal{B}$ covers X and for every $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there is B_3 such that $x \in B_3 \subset B_1 \cap B_2$.
(3) The topology $\mathcal{T}_{\mathcal{B}}$ consists of subsets U in X such that every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.
(4) A subset A of a topological space X is closed if $X - A$ is open.
(5) The closure of A is the intersection of all closed sets containing A.
(6) Let A be a subset of a topological space X. $x \in X$ is a cluster point of A in X if $x \in \overline{A - \{x\}}$.
(7) Let X, Y be topological space. A map $f : X \to Y$ is continuous if $f^{-1}(U)$ is open in X for all open set U in Y. A map $f : X \to Y$ is a homeomorphism if f is a continuous bijection and f^{-1} is also continuous.
(8) A topological space X is Hausdorff if for every distinct points x, y, there are neighborhoods U_x and U_y such that $U_x \cap U_y = \varnothing$.
(9) A topological space X is compact if every open covering has a finite subcovering.
(10) A topological space X is connected if X is not a disjoint union of open sets.
(11) A topological space X is path-connected if every two points can be connected by a path $f : [0,1] \to X$.
(12) A quotient space of a topological space X is given by a space Y such that $f : X \to Y$ is a surjective continuous map and a subset U in Y is open if and only if $\pi^{-1}(U)$ is open in X.
(13) A sequence (x_n) in a topological space X converges to $x \in X$ if for every neighborhood U_x of x, there is N such that $x_n \in U_x$ for all $n > N$.
(14) A metric d on a set X is a map $d : X \times Y \to \mathbb{R}$ such that (i) $d(x, y) \geq 0$ and the equality holds iff $x = y$, (ii) $d(x, y) = d(y, x)$, and (iii) $d(x, z) \leq d(x, y) + d(y, z)$. The metric topology is then the topology generated by ϵ-balls.
(15) For topological spaces X and Y, the product topology on $X \times Y$ is generated by $U \times V$ for all open sets U in X and V in Y.

Theorems:

(1) Let A be a subset of a topological space X. Then $x \in \overline{A}$ if and only if $U_x \cap A \neq \varnothing$ for all neighborhoods U_x of x.
(2) A compact subspace of a Hausdorff space is closed.
(3) A closed interval $[a, b]$ in $\mathbb{R}$ is compact.
(4) A finite product of compact spaces is compact.
(5) The image of a compact space under a continuous map is compact.
(6) Every closed subspace of a compact space is compact.
(7) A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded, i.e. the distance of any two points in A is bounded.
(8) Let X be a metric space and Y a topological space. A map $f : X \to Y$ is continuous if and only if for every convergent sequence (x_n) to x in X, the sequence $(f(x_n))$ in Y converges to $f(x)$.
(9) If X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.
(10) The image of a connected space under a continuous map is connected.
(11) A finite product of connected spaces is connected.
(12) If A is a connected subspace of a topological space, then the closure $\overline{A}$ is also a connected subspace.
(13) If a topological space X is path-connected, then it is connected.