

Introduction to Topology

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1 Topological spaces

A topology is a geometric structure defined on a set. Basically it is given by declaring which subsets are “open” sets. Thus the axioms are the abstraction of the properties that open sets have.

Definition 1.1 (§12 [Mun]). A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that

- (T1) $\emptyset$ and X are in $\mathcal{T}$;
- (T2) Any union of subsets in $\mathcal{T}$ is in $\mathcal{T}$;
- (T3) The finite intersection of subsets in $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ is called a **topological space**. An element of $\mathcal{T}$ is called an **open set**.

Example 1.2. Example 1, 2, 3 on page 76,77 of [Mun]

Example 1.3. Let X be a set.

- **(Discrete topology)** The topology defined by $\mathcal{T} := \mathcal{P}(X)$ is called the **discrete topology** on X .
- **(Finite complement topology)** Define $\mathcal{T}$ to be the collection of all subsets U of X such that $X - U$ either is finite or is all of X . Then $\mathcal{T}$ defines a topology on X , called **finite complement topology** of X .

1.1 Basis of a Topology

Once we define a structure on a set, often we try to understand what the minimum data you need to specify the structure. In many cases, this minimum data is called a basis and we say that the basis generate the structure. The notion of a basis of the structure will help us to describe examples more systematically.

Definition 1.4 (§13 [Mun]). Let X be a set. A **basis of a topology** on X is a collection $\mathcal{B}$ of subsets in X such that

- (B1) For every $x \in X$, there is an element B in $\mathcal{B}$ such that $x \in B$.
- (B2) If $x \in B_1 \cap B_2$ where B_1, B_2 are in $\mathcal{B}$, then there is B_3 in $\mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Lemma 1.5 (Generating of a topology). Let $\mathcal{B}$ be a basis of a topology on X . Define $\mathcal{T}_{\mathcal{B}}$ to be the collection of subsets $U \subset X$ satisfying

- (G1) For every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.

Then $\mathcal{T}_{\mathcal{B}}$ defines a topology on X . Here we assume that $\emptyset$ trivially satisfies the condition, so that $\emptyset \in \mathcal{T}_{\mathcal{B}}$.

Proof. We need to check the three axioms:

- (T1) $\emptyset \in \mathcal{T}_{\mathcal{B}}$ as we assumed. $X \in \mathcal{T}_{\mathcal{B}}$ by (B1).
- (T2) Consider a collection of subsets $U_{\alpha} \in \mathcal{T}_{\mathcal{B}}, \alpha \in J$. We need to show

$$U := \bigcup_{\alpha \in J} U_{\alpha} \in \mathcal{T}_{\mathcal{B}}.$$

By the definition of the union, for each $x \in U$, there is U_{α} such that $x \in U_{\alpha}$. Since $U_{\alpha} \in \mathcal{T}_{\mathcal{B}}$, there is $B \in \mathcal{B}$ such that $x \in B \subset U_{\alpha}$. Since $U_{\alpha} \subset U$, we found $B \in \mathcal{B}$ such that $x \in B \subset U$. Thus $U \in \mathcal{T}_{\mathcal{B}}$.

- (T3) Consider a finite number of subsets $U_1, \dots, U_n \in \mathcal{T}_{\mathcal{B}}$. We need to show that

$$U := \bigcap_{i=1}^n U_i \in \mathcal{T}_{\mathcal{B}}.$$

- Let's just check for two subsets U_1, U_2 first. For each $x \in U_1 \cap U_2$, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. This is because $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$ and $x \in U_1, x \in U_2$. By (B2), there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Now we found $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset U$.
- We can generalize the above proof to n subsets, but let's use **induction** to prove it. This is going to be the induction on the number of subsets.
 - * When $n = 1$, the claim is trivial.
 - * Suppose that the claim is true when we have $n - 1$ subsets, i.e. $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}_{\mathcal{B}}$. Since

$$U = U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$$

and regarding $U' := U_1 \cap \cdots \cap U_{n-1}$, we have two subsets case $U = U' \cap U_n$. By the first arguments, $U \in \mathcal{T}_{\mathcal{B}}$.

Definition 1.6. $\mathcal{T}_{\mathcal{B}}$ is called the **topology generated by a basis** $\mathcal{B}$. On the other hand, if $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ is a basis of a topology such that $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$, then we say $\mathcal{B}$ is a basis of $\mathcal{T}$. Note that $\mathcal{T}$ itself is a basis of the topology $\mathcal{T}$. So there is always a basis for a given topology.

Example 1.7.

- **(Standard Topology of $\mathbb{R}$)** Let $\mathbb{R}$ be the set of all real numbers. Let $\mathcal{B}$ be the collection of all open intervals:

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}.$$

Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the **standard topology** of $\mathbb{R}$.

- Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers, i.e. $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ (cartesian product). Let $\mathcal{B}$ be the collection of cartesian product of open intervals, $(a, b) \times (c, d)$. Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the standard topology of $\mathbb{R}^2$.
- **(Lower limit topology of $\mathbb{R}$)** Consider the collection $\mathcal{B}$ of subsets in $\mathbb{R}$:

$$\mathcal{B} := \left\{ [a, b) := \{x \in \mathbb{R} \mid a \leq x < b\} \mid a, b \in \mathbb{R} \right\}.$$

This is a basis for a topology on $\mathbb{R}$. This topology is called the **lower limit topology**.

The following two lemmata are useful to determine whether a collection $\mathcal{B}$ of open sets in $\mathcal{T}$ is a basis for $\mathcal{T}$ or not.

Remark 1.8. Let $\mathcal{T}$ be a topology on X . If $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{B}$ satisfies (B1) and (B2), it is easy to see that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. This is just because of (G1). If $U \in \mathcal{T}_{\mathcal{B}}$, (G1) is satisfied for U so that $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. Therefore $U = \cup_{x \in U} B_x$. By (T2), $U \in \mathcal{T}$.

Lemma 1.9 (13.1 [Mun]). *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and only if $\mathcal{T}$ is the set of all unions of subsets in $\mathcal{B}$.*

Proof. • $(\Rightarrow)$ Let $\mathcal{T}'$ be the set of all unions of open sets in $\mathcal{B}$. If $U \in \mathcal{T}$, then U satisfies (G1), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. Thus $U = \cup_{x \in U} B_x$. Therefore $U \in \mathcal{T}'$. We proved $\mathcal{T} \subset \mathcal{T}'$. It follows from (T2) that $\mathcal{T}' \subset \mathcal{T}$.

- $(\Leftarrow)$ Since $X \in \mathcal{T}$, $X = \cup_{\alpha} B_{\alpha}$ some union of sets in $\mathcal{B}$. Thus $\forall x \in X, \exists B_{\alpha}$ s.t. $x \in B_{\alpha}$. This proves (B1) for $\mathcal{B}$. If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{T}$ by (T2). Thus $B_1 \cap B_2 = \cup_{\alpha} B_{\alpha}$, $B_{\alpha} \in \mathcal{B}$. So $\forall x \in B_1 \cap B_2, \exists B_{\alpha} \in \mathcal{B}$ s.t. $x \in B_{\alpha}$. This B_{α} plays the role of B_3 in (B2). Thus $\mathcal{B}$ is a basis. Now it makes sense to consider $\mathcal{T}_{\mathcal{B}}$ and we need to show $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$. By the remark, we already know that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. On the other hand, if $U \in \mathcal{T}$, then $U = \cup_{\alpha} B_{\alpha}$, $B_{\alpha} \in \mathcal{B}$. Hence, $\forall x \in U, \exists B_{\alpha}$ such that $x \in B_{\alpha} \subset U$. Thus (G1) is satisfied for U . Thus $U \in \mathcal{T}_{\mathcal{B}}$. This proves $\mathcal{T}_{\mathcal{B}} \supset \mathcal{T}$.

□

Lemma 1.10 (13.2 [Mun]). *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and if any $U \in \mathcal{T}$ satisfies (G1), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.*

Proof.

$\Rightarrow$ Trivial by the definition of $\mathcal{T}_{\mathcal{B}}$.

$\Leftarrow$ X satisfies (G1) so $\mathcal{B}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B} \subset \mathcal{T}$. By (T3), $B_1 \cap B_2 \in \mathcal{T}$. Thus $B_1 \cap B_2$ satisfies (G1). This means (B2) holds for $\mathcal{B}$. Thus $\mathcal{B}$ is a basis. Now the assumption can be rephrased as $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. By the remark above, we already know $\mathcal{T} \supset \mathcal{T}_{\mathcal{B}}$. $\square$

1.2 Comparing Topologies

Definition 1.11. Let $\mathcal{T}, \mathcal{T}'$ be two topologies for a set X . We say $\mathcal{T}'$ is **finer** than $\mathcal{T}$ or $\mathcal{T}$ is coarser than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$. The intuition for this notion is “ $(X, \mathcal{T}')$ has more open subsets to separate two points in X than $(X, \mathcal{T})$ ”.

Lemma 1.12 (13.3). *Let $\mathcal{B}, \mathcal{B}'$ be bases of topologies $\mathcal{T}, \mathcal{T}'$ on X respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T} \Leftrightarrow \forall B \in \mathcal{B}$ and $\forall x \in B, \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.*

Proof. $\Rightarrow$ Since $\mathcal{B} \subset \mathcal{T} \subset \mathcal{T}'$, all subsets in $\mathcal{B}$ satisfies (G1) for $\mathcal{T}'$, which is exactly the statement we wanted to prove. $\Leftarrow$ The LHS says $\mathcal{B} \subset \mathcal{T}'$. We need to show that it implies that any $U \in \mathcal{T}$ satisfies (G1) for $\mathcal{T}'$ too.

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U$$

But

$$\forall B \in \mathcal{B}, \forall x \in B, \exists B' \in \mathcal{B}' \text{ s.t. } x \in B' \subset B.$$

Combining those two,

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B' \in \mathcal{B}' \text{ s.t. } x \in B' \subset B \subset U.$$

$\square$

2 Product topology, Subspace topology, Closed sets, and Limit Points

This week, we explore various way to construct new topological spaces. And then we go on to study limit points. For this end, it is convenient to introduce closed sets and closure of a subset in a given topology.

2.1 The Product Topology on $X \times Y$

The cartesian product of two topological spaces has an induced topology called the product topology. There is also an induced basis for it. Here is the example to keep in mind:

Example 2.1. Recall that the standard topology of $\mathbb{R}^2$ is given by the basis

$$\mathcal{B} := \{(a, b) \times (c, d) \subset \mathbb{R}^2 \mid a < b, c < d\}.$$

Check the axioms (B1) and (B2)!

Definition 2.2 (§15 [Mun]). If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, then the collection $\mathcal{B}$ of subsets of the form $U \times V \subset X \times Y$, $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$ forms a basis of a topology. The topology generated by $\mathcal{B}$ is called **product topology** on $X \times Y$.

Proof.

(B1) Let $(x, y) \in X \times Y$ be an arbitrary element. We need to find a subset in $\mathcal{B}$ containing (x, y) , but since $X \times Y \in \mathcal{B}$, it is obvious.

(B2) For any $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$, the intersection is $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$. So it is obvious again.

□

Example 2.3. The above definition gives a topology on $\mathbb{R}^2$. The following theorem identify this topology with the standard one!

Theorem 2.4 (15.1). If $\mathcal{B}_X$ is a basis of $(X, \mathcal{T}_X)$ and $\mathcal{B}_Y$ is a basis of $(Y, \mathcal{T}_Y)$, then

$$\mathcal{B}_{X \times Y} := \{B \times C \mid B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$$

is a basis of the product topology on $X \times Y$.

Proof. To check $\mathcal{B}_{X \times Y}$, let's use Lemma 1.10 which state that $\mathcal{B}$ is a basis for $\mathcal{T}$ iff for any $U \in \mathcal{T}$ and any $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$. Let $W \in \mathcal{T}$ and $(x, y) \in W$. By the definition of product topology, there are $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x, y) \in U \times V \subset W$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases, there are $B \in \mathcal{B}_X$ and $C \in \mathcal{B}_Y$ such that $x \in B \subset U$ and $y \in C \subset V$. Thus we found $B \times C \in \mathcal{B}_{X \times Y}$ such that $(x, y) \in B \times C \subset W$. □

Example 2.5. The standard topology of $\mathbb{R}^2$ is the product topology of two copies of $\mathbb{R}$ with the standard topology.

Example 2.6. The standard topology of $\mathbb{R}^n$ is given by the basis

$$\mathcal{B} := \{(a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n \mid a_i < b_i\}.$$

Example 2.7. For any $p = (x_0, y_0) \in \mathbb{R}^2$, let $B_{\epsilon, x}$ be the open disk of radius $\epsilon > 0$ centered at p . We can define a topology of $\mathbb{R}^2$ by

$$\mathcal{B}_D := \{D_{\epsilon, x} \mid x \in \mathbb{R}^2, \epsilon \in \mathbb{R}_{>0}\}.$$

The topology defined by $\mathcal{B}$ coincides with the standard topology on $\mathbb{R}^2$.

2.2 The Subspace Topology

A subset of a topological space has a naturally induced topology, called the *subspace topology*. In geometry, the subspace topology is the source of all funky topologies.

Definition 2.8. Let $(X, \mathcal{T})$ be a topological space. Let Y be a subset of X . The collection

$$\mathcal{T}_Y := \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y , called the *subspace topology*.

Lemma 2.9. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then

$$\mathcal{B}_Y := \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis of the subspace topology $\mathcal{T}_Y$ for Y .

Proof. Use Lemma 1.9. Let $V \in \mathcal{T}_Y$, i.e. $V = Y \cap U$ for some $U \in \mathcal{T}$. For every $x \in V$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$ since $\mathcal{B}$ is a basis of $\mathcal{T}$ (Lemma 1.9). Now we found $Y \cap B \in \mathcal{B}_Y$ such that $x \in Y \cap B \subset V$. $\square$ $\square$

2.3 Closed Sets, Closure, Interior, and limit points

Closed sets are nothing but complement of open sets. On the other hand, we can also say that open sets are nothing but complement of closed sets. Thus we can actually use closed sets to define topology, although mathematicians usually use open sets to define topology.

Definition 2.10. Let A be a subset of a topological space $(X, \mathcal{T})$.

- A is a **closed set** of X if $X - A$ is an open set.
- The **closure** $\bar{A}$ of A in X is the intersection of all closed sets of X , containing A .
- The **interior** $\text{Int } A$ of A in X is the union of all open sets of X , contained in A .
- $x \in X$ is a **limit point** of A if $x \in \overline{A - \{x\}}$.

Remark 2.11. It should be easy to see that $A = \text{Int } A$ if A is open and $A = \bar{A}$ if A is closed. How about the other way around? It is easy to see that $A = \text{Int } A$ implies that A is an open set, because any union of open sets is open by (T1). To see that $A = \bar{A}$ implies that A is a closed set, we need to show that any intersection of closed subsets is closed. It is the content of Remark 2.14.

Example 2.12.

- In the standard topology for $\mathbb{R}$, a set of a single element (we say a *point*) is a closed set, because $\mathbb{R} - \{a\}$ is an open set.
- In the discrete topology of a set X , every point is a closed set but also an open set.
- In the lower limit topology, a point is a closed set.

Lemma 2.13 (Interior in terms of closure).

$$\text{Int } A = X - \overline{(X - A)}.$$

Proof. We need to show that $X - \text{Int } A = \overline{X - A}$. Let U denote open sets and C denote closed sets. By definition $\text{Int } A = \bigcup_{U \subset A} U$. Therefore we can do the set theoretic computation:

$$X - \text{Int } A = X - \bigcup_{U \subset A} U = \bigcap_{U \subset A} (X - U) = \bigcap_{C \supset (X - A)} C = \overline{X - A}.$$

$\square$

Remark 2.14 (Defining topology by closed sets). A topology on a set X is given by defining “open sets” of X . Since closed sets are just exactly complement of open sets, it is possible to define topology by giving a collection of closed sets. Let $\mathcal{P}$ be a collection of subsets of X satisfying

(C1) $\emptyset, X \in \mathcal{P}$.

(C2) Any intersection of subsets in $\mathcal{P}$ is also in $\mathcal{P}$.

(C3) Any finite union of subsets in $\mathcal{P}$ is also in $\mathcal{P}$.

Then define $\mathcal{T}$ by

$$\mathcal{T} := \{X - C \mid C \in \mathcal{P}\}$$

is a topology, i.e. it satisfies (T1, 2, 3). On the other hand, if $\mathcal{T}$ is a topology, i.e. the collection of open sets, then

$$\mathcal{P} := \{X - U \mid U \in \mathcal{T}\}$$

satisfies (C1, 2, 3).

Exercise 2.15. Prove the above claims.

Theorem 2.16. Let A be a subset of the topological space $(X, \mathcal{T})$. Let $\mathcal{B}$ be a basis of $\mathcal{T}$.

(a) $x \in \bar{A}$ if and only if every neighborhood U of x intersects with A non-trivially, i.e. $U \cap A \neq \emptyset$.

(b) $x \in \bar{A}$ if and only if every neighborhood $B \in \mathcal{B}$ of x intersects with A non-trivially.

Terminology: U is a **neighborhood** of x if $U \in \mathcal{T}$ and $x \in U$.

Proof.

•
•

□

Example 2.17. The subset $A := \{1/n \mid n = 1, 2, 3, \dots\} \subset \mathbb{R}$ is not closed in the standard topology. To see this, we can apply Theorem 2.16 and Remark 2.11. A is closed if and only if $\bar{A} = A$. So we will show $\bar{A} \neq A$. Observe $0 \notin A$ and let (a, b) be an arbitrary neighborhood of 0 where $a < 0 < b$. Then, no matter how small b is there is n such that $1/n < b$. Thus (a, b) intersects with A non-trivially. Therefore by the theorem, $0 \in \bar{A}$.

Exercise 2.18. The **boundary** ∂A of a subset A of a topological space X is defined by

$$\partial A := \bar{A} - \text{Int } A.$$

From this definition, it follows that $\bar{A}$ is the **disjoint union** of ∂A and $\text{Int } A$, i.e. $\bar{A} = \partial A \cup \text{Int } A$ and $\partial \cap \text{Int } A = \emptyset$.

(a) Find the boundary, the closure and the interior of $(0, 1]$ in $\mathbb{R}$ with the *standard topology*.

(b) Find the boundary, the closure and the interior of $(0, 1]$ in $\mathbb{R}$ with the *finite complement topology*.

(c) Find the boundary, the closure and the interior of $\mathbb{Q}$ in $\mathbb{R}$ with the *standard topology*.

(d) Prove that a subset A is open if and only if $\partial A = \bar{A} - A$.

(e) Prove that $\partial A = \bar{A} \cap \overline{X - A}$.

2.4 Subspace topology and closed sets/closure

Theorem 2.19. *Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a **subspace** of $(X, \mathcal{T})$, i.e. a subset with the subspace topology. Then a subset A of Y is closed in Y if and only if A is an intersection of Y and a closed subset in X .*

Proof. Since an open set in Y is an intersection of Y and an open set in X by definition of subspace topology, this theorem is rather trivial in the perspective of Remark 2.14.

Exercise 2.20. Let Y be a subspace of a topological space $(X, \mathcal{T})$. Prove that, if A is a closed subset of Y and Y is a closed subset in X , then A is a closed subset of X .

Theorem 2.21. *Let Y be a subspace of $(X, \mathcal{T})$ and let A be a subset of Y . The closure of A in Y is $\bar{A} \cap Y$ where $\bar{A}$ is the closure of A in X .*

Proof.

□

References

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