MATH4530: Vocabularies of Basic Set Theory

Tomoo Matsumura

August 20, 2010

1 Sets

"A ***set*** is a collection of ***elements***". We can describe a set by $A = \{a, b, c, d, \cdots\}$ or $A = \{x \mid x \text{ is blah blah blah}\}$.

- $x \in A$ if an element x ***belongs to*** a set A. ($x \notin A$ if an element x does not belong to a set A.)
- $A \subset B$ if a set A is a ***subset*** of a set B, i.e. all elements of A belong to B.
- $A \subsetneq B$ if A is a ***proper subset*** of B, i.e. $A \subset B$ and $A \neq B$.
- The ***complement*** A^c of a subset A of B is defined by $A^c := \{b \in B \mid b \notin A\}$.
- The ***empty set*** $\varnothing$ is defined to be the set with no elements.
- The ***power-set*** $\mathcal{P}(A)$ of a set A is defined to be the set of all subsets of A: To clarify, please note that, if $a \in A$, then $\{a\} \subset A$ and so $\{a\} \in \mathcal{P}(A)$. Writing $\{a\} \in A$ is wrong.
- The ***cartecian product*** $A \times B$ of sets A and B is the set of all *ordered pairs* of elements of A and B:

  $$A \times B := \{(a, b) \mid a \in A, \ \ b \in B\}.$$

  The natural maps $\pi_1 : A \times B \to A, (a, b) \mapsto a$ and $\pi_2 : A \times B \to B, (a, b) \mapsto b$ are called the ***projections (to the 1st and 2nd factors)***.

The ***union***, the ***intersection*** and the ***difference*** of two sets A and B are the sets defined as follows:

- Union: $A \cup B := \{x \mid x \in A$ **or** $x \in B\}$.
- Intersection: $A \cap B := \{x \mid x \in A$ **and** $x \in B\}$.
- Difference: $A - B := \{x \mid x \in A$ **and** $x \notin B\}$. We will use the symbol := to *define* a set.

See page 11 of [Mun] and check the formulas there. We can also define ***arbitrary unions and intersections*** over *any set of sets*: let $\mathcal{I}$ be a set of sets, then

$$\bigcup_{A \in \mathcal{I}} A := \{x \mid x \in A \text{ for at least one } A \in \mathcal{I}\} \quad \text{and} \quad \bigcap_{A \in \mathcal{I}} A := \{x \mid x \in A \text{ for all } A \in \mathcal{I}\}.$$

Caution: *A collection of sets is not a set in general.* For example, if we regard the collection of all sets as a set, we will get into theoretical troubles. In more general framework, such a collection is named a *proper class*. See for example "*Class (set theory)*" in wikipedia. As long as we consider collections of subsets of a set A, it's fine. They are sets, and indeed, subsets of the power-set $\mathcal{P}(A)$.

2 Maps (= Functions)

A ***map*** f from a set A to a set B is an assignment of an element of B to each element of A, denoted by

$$f : A \to B, \;\; a \mapsto f(a) \quad \text{where } a \in A \text{ and } b := f(a) \in B.$$

- The ***composition*** "$g \circ f : A \to C$" of maps $f : A \to B$ and $g : B \to C$ is defined by

$$(g \circ f)(a) := g(f(a)).$$

- The ***image*** of a map $f : A \to B$ is the subset of B given by

$$\mathrm{Im} f := f(A) = \{b \in B \mid \exists a \in A \text{ such that } b = f(a)\}.$$

- Let $f : A \to B$ be a map. The ***inverse image (or preimage)*** $f^{-1}(B')$ of a subset $B' \subset B$ is the subset of A given by

$$f^{-1}(B') := \{a \in A \mid f(a) \in B'\}.$$

- The ***restriction*** of a map $f : A \to B$ to a subset $A' \subset A$ is the map $f|_{A'} : A' \to B$ given by

$$f|_{A'}(a') := f(a') \quad \text{for all } a' \in A'.$$

The ***injectivity, surjectivity*** and ***bijectivity*** of a map $f : A \to B$ is defined as follows:

- f is injective if no two distinct elements of A go to the same element in B under f, i.e. $f(a) = f(a')$ implies $a = a'$
- f is surjective if the image of f coincides with B.
- f is bijective (a one to one correspondence) of f is injective and surjective.

If $f : A \to B$ is bijective, then there is a map $f^{-1} : B \to A$ called the ***inverse*** of f. This map f^{-1} sends $b \in B$ to the unique element $a \in A$ such that $f(a) = b$.

2.1 Compatibility between maps and set theoretical operations $\subset, \cup, \cap$ of subsets

Let $f : X \to Y$ be a map and $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$ subsets. Then f^{-1} preserves unions, intersections, inclusions:

(a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(c) $B_1 \subset B_2$ implies $f^{-1}(B_1) \subset f^{-1}(B_2)$

f preserves unions and intersections only:

(d) $f(A_1 \cup A_2) \supset f(A_1) \cup f(A_2)$.

(e) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$; the equality holds if f is injective.

(f) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

(a), (b), (d) and (e) hold for arbitrary unions and intersections.

3 Relations

A ***relation*** $\sim_R$ on a set A is given by a subset $R \subset A \times A$. If $(x, y) \in R$, then $x \sim_R y$ (x ***is related to*** y).

(I) An ***equivalence relation*** $\sim$ on A is a relation such that

(1) (Reflexivity) $x \sim x$

(2) (Symmetry) $x \sim y$ implies $y \sim x$

(3) (Transitivity) $x \sim y$ and $y \sim z$ implies $x \sim z$

- For an equivalence relation $\sim$ on A and an element $x \in A$, the ***equivalence class*** $[x]$ is a subset of A given by
$$[x] := \{y \in A \mid x \sim y\} \subset A.$$
By the reflexivity $x \sim x$, we have $x \in [x]$.

Lemma 3.1.

(1) $[x] = [y]$ *if and only if* $x \sim y$.

(2) $[x] \cap [y] = \emptyset$ *if and only if* $x \nsim y$.

(3) Let E_1 *and* E_2 *be equivalence classes, then either* $E_1 = E_2$ *or* $E_1 \cap E_2 = \emptyset$.

Remark 3.2. An equivalence relation on A defines a *surjection* $\pi : A \to \mathcal{E}$ where $\mathcal{E}$ is the set of all equivalence classes. On the other hand, if we have a surjection $\pi : A \to \mathcal{E}$ to some set $\mathcal{E}$, then we can define an equivalence relation on A by saying, for each $E_1 \in \mathcal{E}$, all points in $\pi^{-1}(E_1)$ are equivalent to each other. Thus there are one to one correspondence between surjective maps and equivalence classes.

(II) An ***order relation*** (or a simple order, or a linear order) $<$ on A is a relation such that

(1) (Comparability) If $x \neq y$, then $x < y$ or $y < x$.

(2) (Non-reflexivity) If $x < y$, then $x \neq y$.

(3) (Transitivity) If $x < y$ and $y < z$, then $x < z$.

- If $<$ is an order relation, $x \leq y$ means that either $x < y$ or $x = y$.

References

[Mun] Munkres, *Topology* Chapter 1.