

# Backgammon

## Rules

(This section is based on Wikipedia's backgammon entry).

The rules of backgammon are of moderate complexity and can usually be learned quickly. In short, a player tries to get all of his own checkers past those of his opponent and then remove them from the board. The pieces are scattered at first and may be blocked or captured by the opponent. Because the playing time for each individual game is short, backgammon is often played in matches, where, for example, victory is awarded to the player who first wins five games.

## Setup

As figure 1 shows, each side of the board has a track of twelve adjacent spaces, called points, usually represented by long triangles of alternating color. The points are considered to be connected across one edge of the board, forming a continuous track analogous to a horseshoe (but not a circle). In the figure shown, the two areas labeled outer boards are connected across the edge of the board, but the two areas labeled as home boards are not connected. Each player moves her checkers from her opponent's home board toward her own home board. The points are numbered from 1 to 24, with checkers always moving from higher-numbered points to lower-numbered points. The two players move their checkers in opposite directions, so the 1-point for one player is noted as the 24-point for the other player. The game begins with the setup depicted in the figure.

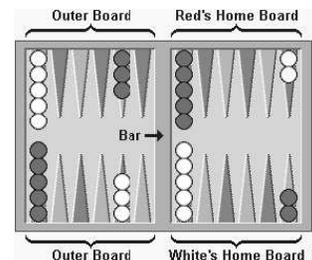


Figure 1: Initial setup of Backgammon

## Movement

At the start of the game, each player rolls one die. The player that rolls the higher number moves first, using the numbers on the two dice already rolled. In the case of a tie, both players roll again. The players then alternate turns, rolling two dice at the beginning of each turn.

After rolling the dice a player must, if possible, move checkers according to the number of points showing on each die. For example, if he rolls a 6 and a 3 (noted as 6-3) he must move one checker six points forward, and another checker three points forward. The dice may be played in either order. The same checker may be moved twice so long as the two moves are distinct: six and then three, or three and then six, but not all nine at once. A checker may land on any point that is either unoccupied or is occupied only by a player's own checkers. It may also land on a point occupied by exactly one opposing checker; such a lone piece is called a blot. In the latter case, the blot has been hit, and is placed in the middle of the board on the bar. A checker may never land on a point occupied by two or more enemy checkers; thus no point is ever occupied by checkers from both players at the same time.

If a player has no legal moves after rolling the dice (because all of the points to which he might move

are occupied by two or more opposing checkers), he must forfeit his turn. However, a player must play both dice if it is possible to do so. If he has a legal move for one die only, he must make that move and then forfeit the use of the other die. If he has a legal move for either die, but not both, he must play the higher number.

If a player rolls two of the same number (doubles) he must play each die twice. For example, upon rolling a 5-5 he must move four checkers forward five spaces each. As before, a checker may be moved multiple times as long as the moves are distinct.

Checkers placed on the bar re-enter the game through the opponent's home field. A roll of 1 allows the checker to enter on the first point of the opponent's home field, a 2 on the second point, etc. A player may not move any other checkers until all of his checkers on the bar have first re-entered the opponent's home field.

When all of a player's checkers are in his home board, he must "bear off", removing the checkers from the board. A roll of 4 may be used to bear off a checker from the fourth point, a 5 from the fifth point, and so on. A die may be used to bear off lower numbered points only if all of the higher numbered points are open. For example, if a player's fourth, fifth, and sixth points are all empty but he has 2 checkers on the remaining points and he rolls a 3-5, he must bear off both checkers from the third point. The two dice may be used in either order, even if this results in not using the "full" value of a die (in some cases this may be strategically advantageous). For example, if you have a checker on the fifth point and two checkers on the first point and you roll a 5-1, you may move the blot from the fifth point to the fourth point and then bear off using the five. If a player has one of her pieces captured during the process of bearing off, that piece must re-enter the game and be moved back into her home board before she can resume bearing off. End of the game

The game ends when one of the players has borne off all of his checkers. If the other player has not borne off any checkers by the time his opponent has borne off all fifteen he has lost a gammon, which counts for double a normal loss (that is, two games toward the match in a game with normal stakes). If he has not borne off any checkers and still has checkers on the bar or in his opponent's home board by the time his opponent has borne off all fifteen, he has lost a backgammon, which counts for triple a normal loss (that is, three games toward the match in a game with normal stakes).

## Basic Strategy

After the opening, experienced players usually choose and combine an array of different strategies. Here are some of them:

- A **running game** consists in moving as quickly as possible around the board. It is most successful when a player is already ahead (i.e. is closer to being ready to bear off).
- A **priming game** consists in building a wall of checkers (with at least two checkers in each point), called a prime, ideally covering six consecutive points. This blocks enemy checkers from passing.
- A **blitz** consists in closing the home board while keeping the opponent on the bar, so that he has difficulty re-entering the game.

## The Doubling Cube

This addition to backgammon is common when played for money. In fact, the doubling cube could be used in any game. The cube has 1, 2, 4, 8, 16, and 32 on its six sides. At the beginning of the game the cube sits in the middle and is on 1. A player who thinks that she is ahead can double, that is, tell the other player to continue and play for 2 points instead of 1. The second player can accept, and play for the two points, or give up on the game and lose 1 point immediately. A player accepting the double has possession of the cube and only he can redouble.

The optimal strategy for the use of the doubling cube depends directly on the probability of winning the game given the current situation. Since this is a very difficult quantity to evaluate, especially because it depends on both the random outcomes of the rolls and on the decisions each player makes, one usually has to make an approximate assessment of this probability, and decide whether to or not to accept a double. In the problems section below we will make the exact calculation of when to double for a simpler game where we can actually calculate these probabilities.

If you are playing a match, another important factor to consider when using the doubling cube is the situation in terms of games won relative to the total number needed to win the match. For example, if you are trailing four games to three in a match to five games you should double at the start of the next game. Losing six games to three is no worse than five games to three (unless you are betting and have attached monetary value to the margin of victory). Therefore you should make sure that if you win the game at hand you will also win the match. Similarly, if you are down three games to one and you decide to accept a double by your opponent, you should re-double immediately.

You can find more information on Backgammon's rules, strategies, and history on the Internet. For instance, you can try Wikipedia: <http://en.wikipedia.org/wiki/Backgammon>.

## Problems

1. Suppose that your opponent's home board is entirely blocked except for the fifth point. If you have a checker in the bar, what is the probability you will be able to enter the game again in your next turn? What is the probability you will be able to enter during the next three turns?
2. Consider the following "flipping pennies" game. At each turn, you and your opponent flip a penny. If the coins are the same, you get them both; if they are different, your opponent gets both. You start with 8 pennies and your opponent starts with 12, and you play until one of you runs out of pennies. What is the probability that you win?
3. Now suppose that there are 100 coins total. How many pennies should you have to offer a double? How many to accept a double? (Hint: Assume that the threshold for offering a double is attained when the expected payoff you get is the same no matter if your opponent accepts or rejects the doubling).

## Solutions

1. To enter in the first turn, you must roll at least one 5. So the probability of not entering on your first turn is  $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$ , meaning that the probability of entering on the first turn is  $1 - \frac{25}{36} = \frac{11}{36}$ . The probability of not entering on any of the first three turns is then  $\left(\frac{25}{36}\right)^3 = \frac{15625}{46656}$ . So the probability of entering during one of the first three turns is  $1 - \frac{15625}{46656} = \frac{31031}{46656}$ , or approximately .665. This type of probability is important to consider when using (or playing against) the blitz strategy described above.
2. Intuitively, as this is a fair game (so the average amount of money you have at the end is the same as the amount with which you started), the answer should be 8/20 (why?). Let's see that this is correct. Call  $p$  the probability that you win. Then the probability that your opponent wins is  $1 - p$  (observe that we are supposing here that there is no possible tie; this is true and can be proved). If you win, you end up with 20 pennies; if you lose, you get 0. Thus the average amount of money you end up with is  $20p + 0 \cdot (1 - p) = 20p$  pennies. Since this is a fair game, this average amount has to be 8 pennies. Therefore,  $20p = 8$  so  $p = \frac{8}{20} = 0.4$ .
3. The idea we will use is to assume that the threshold for offering a double is attained when the expected payoff you get is the same no matter if your opponent accepts or rejects the double. (This means, intuitively, that having more coins than the threshold implies that you would be better off if the opponent accepts the double, so you should try offering it).

We will include doubling in this game in the following way: when a player gets all the coins, he wins 1 point, while the loser loses 1 point. If the doubling cube is at 2, he gets 2 points, and the loser loses 2, and so on.

Let's call  $d$  the minimal amount of coins you should have before doubling. This threshold  $d$  should be such that the average payoff you get if your opponent rejects the doubling is the same as the payoff you get if she accepts it. We will suppose that the current bet is 1 (this does not make any difference). By the same argument used to solve problem 2 above, when you have  $d$  coins, the probability that you win is  $\frac{d}{100}$ , and your expected payoff is exactly:

$$\frac{d}{100} - \frac{100 - d}{100} = \frac{2d - 100}{100}$$

(since here we are considering that upon winning you win 1 point, and upon losing you lose 1 point).

Let's call  $q$  the probability that starting with  $d$  coins (where  $d > 50$ ) you win before ever having  $100 - d$  coins. If this event happens, you win 1 point (forget the doubling for a while). If the opposite happens, then your expected return drops to  $\frac{100 - 2d}{100}$ . However, since you now have  $d$  coins, the current expected payoff is  $\frac{2d - 100}{100}$ . Therefore we can calculate the expected value in two different ways. Setting these expressions equal yields:

$$q + \frac{100 - 2d}{100} \cdot (1 - q) = \frac{2d - 100}{100}$$

Solving this equation gives  $q = \frac{2d - 100}{d}$ .

Now, when you have the threshold value  $d$  of coins your expected payoff is 1, since that is what you would get if your opponent rejects the doubling. If he accepts the double you may win or lose the bet. The event of winning the bet before ever having  $100 - d$  coins has probability  $q$ , and that gives you a payoff of 2. If you reach  $100 - d$  coins, your opponent will redouble, so your expected payoff will be -2 regardless of if you accept or reject the redoubling. Therefore, we must solve the equation

$$1 = 2q - 2(1 - q) = 4q - 2$$

Replacing the expression for  $q$  in terms of  $d$  yields

$$\frac{3}{4} = \frac{2d - 100}{d}$$

or

$$3d = 8d - 400$$

Solving gives  $d = \frac{400}{5}$ , so you should have at least 80 coins before offering a double.