"Straight" and "Angle" on Non-Planar Surfaces

{The idea is not to do technical details but rather to apply the meanings of "straight" and "angle" that the students have already agreed upon while studying lines & angles on the plane.}

T: Previously, we explored meanings of "straight" on the plane. Do you remember?

S1: Sure, we agreed that

1. A straight line has no turning left or right at any point.
2. A straight line contains the shortest distance between any two of its points.
3. A straight line has mirror symmetry (left side and right side are the same).
4. A straight line has half-turn (180°) symmetry about every point on the line.
5. A straight line has translation symmetry along itself.

1. Spheres

The students are divided into groups of 3 or 4 and each group is supplied with Styrofoam balls (at least 4 inches in diameter), or a beach ball or similar smooth sphere – soccer balls or volleyballs will work but basketballs tend not to be very smooth. Each group should also have push pins (if using Styrofoam), some lengths of ribbon (long enough to fit around the sphere and be tied), toys cars (with non-steerable wheels), yarn (long enough to fit around the sphere and be tied), and a transparent disk (about 2 inches -- small enough to nearly lay flat on the sphere) with a small hole punched in the center with a straight line segment drawn on it thru the center from edge to edge.

T: Imagine yourself to be a bug crawling around on a sphere. (This bug can neither fly nor burrow into the sphere.) The bug's universe is just the surface; it never leaves it.

S3: Oh, you mean like an ant crawling on a beach ball?

T: Yes, that's right. And notice that the ant appears not to be affected by gravity or anything off the surface.

S3: Yeah, the ants seem perfectly happy walking upside down on the bottom of the ball.

T: Now imagine you are this bug (or ant) on the surface of a sphere. What would you experience as "straight"? How can you convince yourself of this? Use the meanings of straight that we agreed on in Section I.

S2: But wouldn't a bug only experience straight if it burrowed thru a straight tunnel?

T: Yes, this would probably be true if the bug was thinking of navigating in the 3-space that contains the sphere; but imagine that the bug is bound to the surface of the sphere, much the same way that we are bound to the surface of the earth.

The important thing here is to **think in terms of the surface of the sphere, not the solid 3-dimensional ball**. Clearly, to us outside of the sphere any path on the sphere will appear curved (not-straight). But what will be the experience of the bug who is only experiencing the surface of the sphere?

{***Note to teacher:* A good example of how this type of thinking works is to look at an insect called a water strider. The water strider walks on the surface of a pond and has a very 2-dimensional perception of the world around it — to the water strider, there is no up or down; its whole world consists of the 2-dimensional plane of the water. The water strider is very sensitive to motion and vibration on the water's surface, but it can be approached from above or below without its knowledge. Hungry birds and fish take advantage of this fact. This is the type of thinking needed to visualize adequately properties of straight lines on the sphere. For more discussion of water striders and other animals with their own varieties of intrinsic observations, see the delightful book "The View from the Oak", by Judith and Herbert Kohl [NA: Kohl and Kohl].** }

{The teacher could try starting the following activity as individual explorations but it seems to work better in small groups because more than two hands are helpful when working with the spheres.}

Group Activity
Imagine yourself to be a bug crawling around on a sphere. (*This bug can neither fly nor burrow into the sphere.*) *The bug's universe is just the surface*; *it never leaves it. What is "straight" for this bug*? *What will the bug see or experience as straight*? *How can you convince yourself of this*? *Think about your experiences with straight lines on the plane.*

What would angles be like for the bug. How would the bug experience perpendicular lines, triangles, squares, circles, and so forth? What in the bug's experience would be like on the plane and what would be different?

Worksheet 1:

Team Work

Now each GROUP explores the following activities and questions using their beachball, yarn, ribbons, and transparent disk. They then come up with some answers that are later shared with the whole class. The report-backs to the whole class could come after each mini-activity or at the end. The teacher should make this call depending on the students in the class and the length of time in a class period. In each case connect what you do here with what you remember about lines and angles in the place.

A. Mark two points on the sphere. Stretch a piece of yarn between the two points.

What do you notice? Where does it seem to fit best?	**Related to which of the 5 properties of straightness?**

B. Take a ribbon and try to "lay it flat" on the sphere.

What do you notice? Along which paths does the ribbon lie flat?	**Related to which of the 5 properties of straightness**
What does this have to do with mirror symmetry?	**Related to which of the 5 properties of straightness?**

C. Imagine the bug walking along a path on the sphere.

On what paths will the bug's right and left legs be moving the same?	**Related to which of the 5 properties of straightness?**

D. Take a small toy car with its wheels fixed to parallel axes so that, on a plane, it rolls along a straight line. Try rolling this toy car on the sphere.

What do you notice? What curves does the car follow?	**Related to which of the 5 properties of straightness?**

E. Use your transparent disk to explore the symmetries of curves on the sphere.

What do you notice? Which curves have symmetry?	**Related to which of the 5 properties of straightness?**

Class Work

The small groups of students report back to the whole group about what they observed and the whole class agrees on statements about "straight" on a sphere. Hopefully, the class will notice that certain circles on the sphere seem to have the same five properties stated at the beginning of the section for straight lines on the plane.

T: What did you observe?

S1: The biggest circles on the sphere (the ones that divide the sphere is half) seem to satisfy all of the same five properties that we talked about for straight lines on the plane.

T: Good. Mathematicians call these circles **great circles** (dictionary term) and define them by saying: Great circles are those circles on a sphere that are the intersection of the sphere with a plane through the center of the sphere.

S2: But I thought that the bug can't see such planes because they would be off the surface.

T: That is correct. This description of great circles is for us who are viewing the great circles on the sphere from the outside. How does the bug view the great circles?

S3: As a straight line! The bug would think it is going straight.

T: Mathematicians call our point-of-view from the outside "**extrinsic**" and the bug's view of only the surface as "**intrinsic**". So we say that the great circles on a sphere are *extrinsically* (from our outside point-of-view) *curved* but they are *intrinsically* (from the bug point-of-view) *straight.*

Writing Exercise

Write a paragraph (or more) about: Why are you convinced that the great circles on a sphere are intrinsically straight and why no other circles on the sphere are intrinsically straight. Or, if you are not convinced, then what is unanswered for you? Take what you learned from Worksheet 1 (and/or other things) and write what is <u>most convincing to you</u>.

Each student's writing is shared with the class (by reading and/or posting?) and students make constructive comments on each other's papers.

Locate a globe of the earth for the classroom situated so all can see it and have a class discussion about which paths on the globe are intrinsically straight and which are not. Why is this important for navigating airplanes and ocean-going ships?

S3: This is like the equator on the earth – this must be a great circle.

T: Correct. Can you show the other students this on a globe? Do you see other circles on the globe that are great circles?

M: Paths that are intrinsically straight on a sphere (or other surfaces) are called ***geodesics*** (dictionary term).

S1: So on the sphere arcs of great circles are geodesics.

T: Find the shortest path on the Earth from your city to Cairo, Egypt. Then try this for other pairs of cities.

T: Here is what we have agreed on:

On a sphere:

- Great circles are the intrinsically straight (geodesics) paths and other circles are not.
- A great circle contains the shortest distance between any two of its points.
- A great circle has mirror symmetry.
- A great circle has half-turn symmetry about every point on it.
- A great circle has translation symmetry along itself.

{Teacher notes: Some students may find the above difficult, though this did not happen in the class at Lanier High School. (It often does happen with math majors at Cornell!) If it happens with your students you might try the dialogue:
M: It is natural for you to have some difficulty experiencing straightness on the sphere; it is likely that you will start out looking at spheres and the curves on spheres as 3-dimensional objects.

S2: Yeah. They look three-dimensional.

M: However, intrinsically, from the bug point-of-view they are two-dimensional because as the bug walks along the surface of a sphere it has only two choices of direction -- left or right and forward or back. }

{Teacher note: The following could be included IF your students bring it up:

M: Notice that, on a sphere, straight lines are intrinsic circles (points on the surface situated at a fixed distance along the surface from a given point on the surface) — special circles whose circumferences are straight! Note that the equator on the earth is an intrinsic circle with two intrinsic centers: the North Pole and the South Pole. In fact, any circle (such as a latitude circle) on a sphere has two intrinsic centers. }

ANGLES on the sphere:

***Review what you learned about angles on a plane*. Make a list of the statements about angles that you accepted at that time.**

Worksheet 2:

Team Work

Mark a point on the sphere and take your transparent disk and yarn. Place the disk on the sphere so the hole in the center is at the point and hold it there with a pencil point thru the hole. Tie a piece of yarn around the point of the pencil and rotate it around the point.

Which meanings of "angle" and "angle measure" that you explored in Section II apply on the sphere?	**Why?**
Do some accepted statements not apply?	**Why not?**

Class Work	**Each group reports back to the whole class and the class discusses and agrees on statements about angles, rotations, and angle measure on a sphere – as much as possible referring back to the discussions of angles on the plane.**

Team Work

Use your yarn to designate two intersecting intrinsic straight line segments on the sphere. Explore one after the other each of the proofs and associated activities of the Vertical Angle Theorem at the end of L&A-II and in L&A-IV. What do you find? Do these proof apply to intrinsic straight lines on a sphere? Use your transparent disk and yarn as appropriate.

Proof of VAT using half-turn symmetry of lines. Does this apply on the sphere?
Proof of VAT using angle as rotation. Does this apply on the sphere?
Proof of VAT involving supplementary angles. Does this apply on the sphere?

Each group reports back to the whole class. The class discusses and agrees on statements.

Team Work

Use your yarn to mark intrinsically straightline segments on the sphere holding the ends with tape. Mark several segments and compare what you see with what happens with straight lines on the plane. Write down what you notice and how it compares with straight lines on the plane.

Draw a large triangle of the sphere	**What do you notice about the angles?**
Draw two lines that are parallel transports along a transversal	**What do you notice?**
Draw another figure	**How does it compare to figures on plane?**
Draw another figure	**How does it compare to figures on plane?**

Each group reports back to the whole class. The class discusses and agrees on statements.

{Note to teacher: Here is a list of some of the statements that the groups and class may come up with:

1. **Any two great circles intersect, but parallel straight lines on the plane do not intersect.**
2. **Great circles are finite and come back on themselves, but straight lines go on forever on the plane.**
3. **Even though great circles are finite you can walk along the great circle forever in either direction -- this is the same as for straight lines on the plane except that on the sphere you are retracing your steps.**
4. **The Vertical Angle Theorem holds, the same as on the plane.**
5. **Some great circles are parallel transports along a transversal (another great circle). Example: The longitude lines on the earth are parallel transports of each other along the equator. On the plane parallel transported lines never intersect.**
6. **Three great circles can form a triangle (in fact 8 triangles!).**
7. **Some triangles on the sphere have angle sum more than 180 degrees. Example: there is a triangle with three right angles. On the plane the sum of the angles of a triangle is always 180 degrees.**
8. **Not all arcs in a great circle are the shortest distance between their endpoints.**

It is not important that the students come up with or agree on all of these statements. It is also OK if they disagree with each other or if they make conjectures/questions that no one knows how to prove (for example: Is there any triangle on the sphere which has angle sum equal to 180 degrees?). Disagreements and open conjectures and questions can be a good source for individual or small group projects and papers. It would be a profitable experience if the students debated with each other (orally and in writing). }

{The following subsection can be left out at the discretion of the teacher.}

2. Hyperbolic Soccer Balls.

Each group is given sheets of paper tessellated by hexagons and have at least one soccer ball (made from hexagons and pentagons) in the classroom.

T: As you see, hexagons (six sides) can fill up the plane. This is called a **tessellation** of the plane. Note that in this tessellation every hexagon is surround by six other hexagons. Could we instead have a tessellation with pentagons (five sides) each surrounded by 5 hexagons?

S: Sure that looks like a soccer ball!

T: Yes, let's look at the soccer ball and see that each black pentagon is surrounded by five white hexagons.

At the teacher's option, each group could build a paper soccer ball using hexagons and pentagons. This can be done using the template on the next page and leaving the pentagonal holes empty.

T: Now, what would happen if we started with a seven-sided polygon (called a *heptagon*) and surrounded it with seven hexagons?

S1: I can't imagine it.

S2: It would be weird.

S3: Can we try to build one?

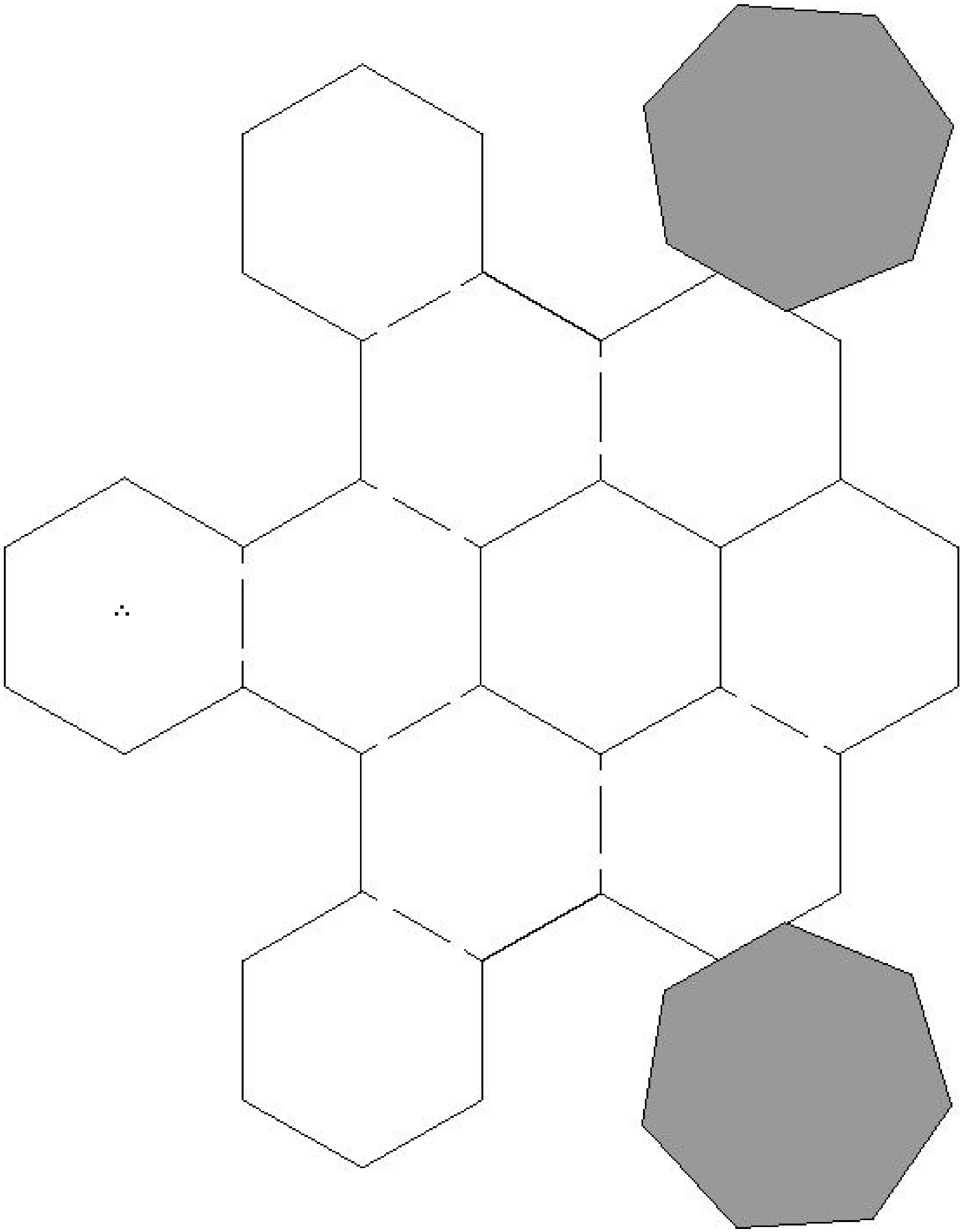
T: Yes, let's do that.

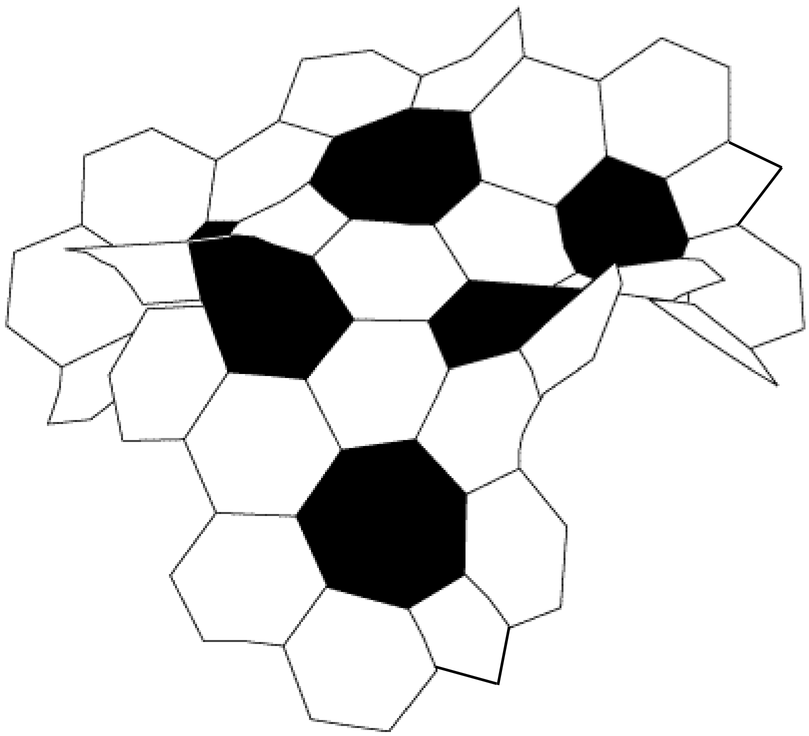
Each group is given sheets of templates (next page) – one for each student. The students will cut out the hexagons and heptagons according to the directions:

Cut along the solid lines and not along the dashed lines. Save the hexagon cut out from the center and replace it with a heptagon. Then use the removed hexagon to complete the surrounding of the heptagon. Put three of these together by overlapping the hexagons with the three dots. Tape the remaining heptagons into the surface in such a way that every heptagon is surrounded by seven hexagons and each hexagon is surrounded by three heptagons alternating with three hexagons.

The finished surface should look like the picture on the page after the template.

Template for hyperbolic soccer ball.





This is a picture of a finished hyperbolic soccer ball. This can be continued by adding more hexagons and heptagons in such a way that each heptagon is surrounded by seven hexagons and each hexagons surrounded by three heptagons interweaved with three hexagons.

Team Work	**Use ribbons to explore intrinsic straight lines (geodesics) on the hyperbolic soccer ball. You can also carefully stretch the surface and yarn between the two points. Or use the small toy car. What do you find?**

Can you find two geodesics that do not intersect?
Can you make a large triangle (with geodesic sides)? What can you say about the sum of the angles of this triangle?
Compare intrinsic straight lines on the hyperbolic soccer ball to those on the plane and to those on the sphere.
What else to you notice?

Each group reports back to the whole class.