# Solutions

## The Binomial Model

- Activity 1

  a) Suppose that you short-sell €1 today. If you wait two days, because the interest rate is %10 the day after tomorrow you will receive $\$(1.5*(1.1)^2) = \$1.815$. If the price of the Euro goes up these two days then you pay your debt back at the end of the month and your total profit is 0. If the price of the Euro goes down one day but up the other one, once you pay back the Euro that you short-sold your profit will be equal to \$(1.815-1.320)=\$0.495. If the price of the Euro goes down both days then after paying back your debt your profit will be equal to \$(1.815-0.96)=\$0.855 with probability $(0.5)^2 = 0.25$.

  b) Suppose that there is only one day left in the month. If you short-sell €N today, tomorrow you will receive \$(1.5*1.05)N=\$1.575N. If the price of the Euro goes up you would lose \$(1.65-1.575)N=\$0.075N with probability 0.5. Hence your strategy of short-selling Euros in the beginning is not riskless.

     Now suppose that you borrow \$1.5N and buy €N today. Tomorrow you will owe \$(1.5*1.05)N. If the price of the Euro goes down after selling your €N tomorrow you will lose \$(1.575-1.2)N=\$0.375N with probability 0.5. Hence your strategy is not free of risk. It seems then that under these circumstances there is no way of beating the market. The next two section will prove this fact more formally.

- Activity 2

  a) By short-selling 1 gallon of gasoline on January $1^{st}$, the profit on January $2^{nd}$ will be equal to either \$(4.8-4.4)=\$0.4 if the price rises or \$(4.8-3.6)=\$1.2 if the price drops.

  b) By short-selling gasoline on January $1^{st}$ and waiting unitil January $4^{th}$ the profit will always be positive (see figure below).

  c) Suppose we are facing the situation described in a). If you short-sell gasoline and the price rises you will lose money on January $2^{nd}$ with probability 0.5. If you buy gasoline and sell it on January $2^{nd}$, you will lose money if the price drops (see b) of Activity 1).

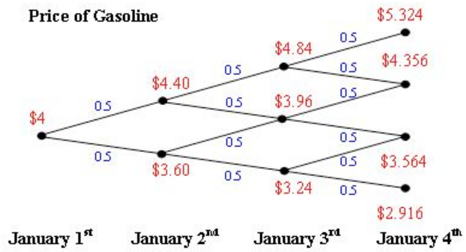


Figure 1: Activity 2 b)

# Arbitrage Opportunities

a) By the formula given at the end of the lesson we deduce that

$$x_2^u = x_1 + (y_1 - y_2^u) * 10 * 1.01$$
$$x_2^d = x_1 + (y_1 - y_2^d) * 10 * 0.97.$$

b) The value of the initial portfolio $(x_1, y_1)$ is $x_1+10y_1$. Let $V^{uu}, V^{ud}, V^{du}, V^{dd}$ be the final values of the portfolio $(x_2, y_2)$ when the price rises both days, rises the first day and drops the second day, drops the first day and rises the second day and drops both days, respectively. By using a) and the definition of the value of a portfolio we obtain that

$$\begin{aligned}
V^{uu} &= x_2^u + y_2^u * 10 * (1.01)^2 = x_1 + 10 * 1.01 * y_1 + 0.101 y_2^u \\
V^{ud} &= x_2^u + y_2^u * 10 * 1.01 * 0.97 = x_1 + 10 * 1.01 * y_1 - 0.303 y_2^u \\
V^{uu} &= x_2^d + y_2^d * 10 * 1.01 * 0.97 = x_1 + 10 * 0.97 * y_1 + 0.097 y_2^d \\
V^{uu} &= x_2^d + y_2^d * 10 * (0.97)^2 = x_1 + 10 * 0.97 * y_1 - 0.291 y_2^d.
\end{aligned}$$

c) If we assume that $x_1 + 10y_1 \leq 0$ and that $V^{uu}, V^{ud}, V^{du}, V^{dd} \geq 0$, by b) we conclude that

$$0.1y_1 + 0.101y_2^u \geq 0 \tag{1}$$
$$0.1y_1 - 0.303y_2^u \geq 0 \tag{2}$$
$$-0.3y_1 + 0.097y_2^u \geq 0 \tag{3}$$
$$-0.3y_1 - 0.291y_2^u \geq 0. \tag{4}$$

By (1) and (2) we conclude that $y_1 \geq 0$. By (3) and (4) we deduce that $y_1 \leq 0$. Therefore $y_1 = 0$ and (1)-(4) imply that $y_2^u$ and $y_2^d$ are 0 as well. Hence, none of the inequalities is strict and the market is arbitrage-free.

# The First Fundamental Theorem of Asset Pricing

- Activity 1

  a) Since for any time $t$, $X_{t+1}$ depends on the past only through $X_t$

  $$\begin{aligned} E[X_{t+1}|X_0 = x_{i_0}^0, \dots, X_t = x_{i_t}^t] &= E[X_{t+1}|X_t = x_{i_t}^t] \\ &= (x_{i_t}^t + 1) * 0.5 + (x_{i_t}^t - 1) * 0.5 \\ &= x_{i_t}^t. \end{aligned}$$

  b) As we saw in the lesson $E^P[S_2|S_1 = 9.7] = 9.5254 \neq 9.7$. Hence $S$ is not a martingale under $P$.

- Activity 2

  a) The states of the market depend on the behavior of the process $R$. Hence the main events are of the form $\{R_{t_1} = r_{t_1}, \dots, R_{t_n} = r_{t_n}\}$. Since we are assuming that $R_t$ and $R_{t+1}$ are independent for all $t$, we have that $Q[R_{t_1} = r_{t_1}, \dots, R_{t_n} = r_{t_n}] = 0$ if and only if $Q[R_{t_i} = r_{t_i}] = 0$ for some $i$ if and only if $q = 0$ or $q = 1$. A similar argument works for $P$. This explains why if $p$ is strictly between 0 and 1, under our hypothesis, $Q$ is equivalent to $P$ if and only if $q$ is strictly between 0 and 1.

  b) If we let $Q^*[R_1 = b] = q^*$, then

  $$E^{Q^*}\left[\frac{S_1}{(1+r)}\Big|S_0 = x\right] = x,$$

  if and only if

  $$\frac{x(1+b)}{(1+r)}q^* + \frac{x(1+a)}{(1+r)}(1-q^*) = x,$$

  if and only if

  $$q^* = \frac{r-a}{b-a}.$$

  c) In the multi-period case we have that $X$ is a martingale under $Q^*$ if and only if for all $t$

  $$E^{Q^*}\left[\frac{S_{t+1}}{(1+r)^{t+1}}\Big|X_t = \frac{x}{(1+r)^t}\right] = \frac{x}{(1+r)^t},$$

  if and only if for all $t$

$\frac{x(1+b)}{(1+r)} Q^* \left[ R_{t+1} = b \middle| X_t = \frac{x}{(1+r)^t} \right] + \frac{x(1+a)}{(1+r)} Q^* \left[ R_{t+1} = a \middle| X_t = \frac{x}{(1+r)^t} \right] = x,$

if and only if for all $t$

$$Q^* \left[ R_{t+1} = b \middle| X_t = \frac{x}{(1+r)^t} \right] = \frac{r-a}{b-a},$$

if and only if for all $t$, $R_{t+1}$ is independent of $R_t$ and

$$Q^*[R_t = b] = \frac{r-a}{b-a}.$$

d) By a)-c) $Q^*$ is a probability measure equivalent to $P$ such that $X$ is a $Q^*$-martingale if and only if $q^* = \frac{r-a}{b-a}$ is strictly between 0 and 1. This holds if and only if $a < r < b$. The conclusion follows then from the First Fundamental Theorem of Asset Pricing.

e) If $p$ is either 0 or 1 the market is deterministic and there is no arbitrage if and only if $p = 1$ and $r = b$ or $p = 0$ and $r = a$.

f) In Examples 1 and 2 $p = 0.5$, $b = 0.1$, $a = -0.2$ and $r = 0.1$. $b = r$ and the market is not arbitrage-free. If $r = 0.05$ then the market is arbitrage-free (see Activity 1 b) of The Binomial Model lesson). In Example 3 $p = 0.3$, $b = 0.01$, $a = -0.03$ and $r = 0.02$. $r > b$ and the market is not arbitrage-free. In the last Example if $r = 0$ the market is arbitrage-free (see Activities of the Arbitrage Opportunities lesson).

# Financial Derivatives

a) The payoff of a straddle is $|S_T - S_0|$, the absolute value of $S_T - S_0$. In our model under the risk-neutral probability measure the possible discounted payoffs of a straddle with maturity of two days are $\$(10 * (1.01)^2 - 10) = \$0.201$ with probability $(0.75)^2$, $\$(10 - 10 * 1.01 * 0.97) = \$0.203$ with probability $0.75 * 0.25$ and $\$(10 - 10 * (0.97)^2) = \$0.591$ with probability $(0.25)^2$. Hence,

$$\pi^{straddle} = 0.201 * (0.75)^2 + 2 * 0.203 * 0.75 * 0.25 + 0.591 * (0.25)^2 \approx 0.23.$$

b) Let $S$ be the price of the text-book. We have $S_0 = 80$, $a = -0.1$, $b = 0.2$, $p = 0.5$ and $r = 0$. The risk neutral measure is given by $Q^*[R_t = b] = q^* = \frac{1}{3}$. The possible discounted payoffs of a European call option with maturity of two days and strike \$80 are \$35.2 with probability $\frac{1}{9}$, \$6.4 with probability $\frac{2}{9}$ and \$0 with probability $\frac{4}{9}$. Hence

$$\pi^{call} = \frac{35.2}{9} + 2\frac{12.8}{9} \approx 6.75.$$

c) In this case the possible payoffs are \$80 with probability $\frac{5}{9}$ and \$64.8 with probability $\frac{4}{9}$. Hence

$$\pi^{dc} = \frac{400}{9} + \frac{259.2}{9} \approx 73.24.$$

# Hedging

a) The model is arbitrage-free because $a < r < b$. We have to solve the following system of linear equations for $x_1$ and $y_1$

$$(1.05)x_1 + 1.65y_1 = 0$$
$$(1.05)x_1 + 1.2y_2 = 0.3.$$

The unique solution to this system is $y_1 = -\frac{2}{3}$, $x_1 = \frac{1.1}{1.05}$. This corresponds to short-selling €$\frac{2}{3}$ and holding \$$\frac{1.1}{1.05}$. The value of this portfolio at time 0 is \$ $\left(\frac{1.1}{1.05} - \frac{2}{3}1.5\right)$, the price of the put option.

b) We have to solve the following system of linear equations for $x_1$ and $y_1$

$$x_1 + 10.1y_1 = 0.1$$
$$x_1 + 9.7y_2 = 0.3.$$

The unique solution to this system is $y_1 = -\frac{1}{2}$, $x_1 = 5.15$. This corresponds to short-selling $\frac{1}{2}$ of a share of the stock and holding \$5.15. The value of this portfolio at time 0 is \$ $(5.15 - 5) =$ \$0.15, the price of the straddle.

c) This market is arbitrage-free and under the risk-neutral probability measure the probability that the price rises is $\frac{5}{6}$.

c) To hedge the position you have to solve the following system of linear equations for $x_1$ and $y_1$

$$x_1 + 33y_1 = 3$$
$$x_1 + 15y_1 = 0.$$

The unique solution to this system is $y_1 = \frac{3}{18}$, $x_1 = -\frac{45}{18}$. This corresponds to buying $\frac{3}{18}$ bottles of wine and borrowing \$$\frac{45}{18}$. The value of this portfolio at time 0 is \$2.5, the price of the contract. Of course, buying fractions of a bottle is very unrealistic. Hedging and Pricing under constraints could become very complex.