Solutions: Module 2

Dividend payments

- Activity 1

  This is not an arbitrage opportunity because if the price of the Euro goes up, after considering dividend payments the value of the strategy tomorrow is
  $$-10 * (1.01)^2 + 10 * 1.02 = -0.001.$$
  Therefore this strategy does not represent a riskless investment. The risk neutral probability q^* is given by
  $$q^* = \frac{r - d'}{u' - d'} = \frac{0.02 + 0.0203}{0.0201 + 0.0203}.$$

- Activity 2

  If the interest rate in Europe is 31.25% the new rates of return are
  $$d' = 0.8 * 1.3125 - 1 = 0.05; \; u' = 1.1 * 1.325 - 1 = 0.44375.$$
  The condition $d' < r < u'$ does not hold anymore and hence the market is not arbitrage free. An arbitrage opportunity would be to buy one Euro and borrow 1.5 dollars. Even if the price of the Euro goes down, it would be enough to cover the debt of \$1.575.

- Activity 3

  a) The conditions on s are given by the inequalities
  $$(1 + d)(1 + s) - 1 < r < (1 + u)(1 + s) - 1.$$
  Which is equivalent to
  $$\frac{1 + r}{1 + u} - 1 < s < \frac{1 + r}{1 + d} - 1.$$

  b) We would have
  $$q^* = \frac{r - d'}{u' - d'} = \frac{1 + r - (1 + d)(1 + s)}{(1 + s)(u - d)}.$$

c) The price would be equal to

$$\pi = \frac{1}{1+r}(S_0(1+u)q^* + S_0(1+d)(1-q^*)) = \frac{S_0}{1+s}.$$

With no dividend payments the fair price would be S_0. Hence the dividend payments make the price of this claim smaller, since by purchasing this contract the buyer is not receiving the dividends paid between today and tomorrow.

Families of risky assets

- Activity 1

In this case if the prices go down the value of the strategy tomorrow is

$$1.36 - 1.2 * 1.05 - 0.105 = -0.005.$$

Therefore the investment is no longer a riskless strategy.

- Activity 2

a) If $r = 0.05$ then the quotients

$$\frac{r - d^i}{u^i - d^i},$$

for $i =$€, £are both equal to $\frac{2}{3}$. If the interest rates on the Euro and pound are s and t respectively then the model is arbitrage free if and only if

$$0 < \frac{1.05 - 0.8(1+s)}{0.3(1+s)} = \frac{1.05 - 0.85(1+t)}{0.3(1+t)} < 1.$$

b) In this case $d^{GM} = -0.03$, $u^{GM} = 0.01$, $d^F = -0.02$, $u^F = 0.02$ and $r = 0$. We have that

$$0 < \frac{r - d^F}{u^F - d^F} = \frac{1}{2} < \frac{3}{4} = \frac{r - d^{GM}}{u^{GM} - d^{GM}} < 1.$$

Therefore the model is not arbitrage free.

- Activity 3

a) For instance the first day the rates of return on the call option are $u^{call} = \frac{0.4-0.3181875}{0.3181875}$ and $d^{call} = \frac{0.07275-0.3181875}{0.3181875}$. We have that $\frac{r-d^{call}}{u^{call}-d^{call}} = \frac{3}{4}$. Then the condition holds for the first time period. The other verifications are analogous.

b) The price of the put option at time 0 is 0.0181875. If the prices go up the price tomorrow is 0. If the prices go down the price tomorrow is 0.291.

Time dependent interest rates and rates of return

- Activity 1

  a) Consider the following strategy. Do not trade today. Tomorrow short one share of GM and lend the corresponding price in dollars, either 10.1 or 9.7. This strategy is self-financing and an arbitrage opportunity.

  b) In this case the strategy above is no longer an arbitrage opportunity because if the price of GM goes up tomorrow the value of the strategy is negative. Hence, we conjecture that in this case the market is arbitrage-free. See Activity 2 a) below.

- Activity 2

  a) In activity 1 a) the rates of return on the stock tomorrow, $d_1 = -0.03$ and $u_1 = 0.01$, and the interest rate tomorrow $r_1 = 0.01$ do not verify the condition $d_1 < r_1 < u_1$, and hence the market is not arbitrage free. In activity 2 b) the new rates of return tomorrow are $d_1' = -0.0203$ and $u_1' = 0.0201$. They verify the condition $d_1' < r_1 < u_1'$ and the market is arbitrage free.

  b) Assume that there are N assets with time dependent rates of return $d_0^i, \dots, d_{T-1}^i$ and $u_0^i, \dots, u_{T-1}^i$ for $i = 1, \dots, N$. The interest rates are $r_0, \dots, r_{T-1}$. The prices of the assets move up and down simultaneously. Then the market is arbitrage free if and only if for any t between 0 and $t-1$ and i, j between 1 and N

  $$0 < \frac{r_t - d_t^i}{u_t^i - d_t^i} = \frac{r_t - d_t^j}{u_t^j - d_t^j} < 1.$$

- Activity 3

  a) The prices of the call option tomorrow are either

  $$\pi_1^{call} = \frac{1}{1.01}(0.75 * 10(1.01^2 - 0.97) + 0.25 * 10 * 0.97 * 0.01) = \frac{0.4}{1.01},$$

  if the price goes up today or

  $$\pi_1^{call} = \frac{1}{1.01}(0.75 * 10 * 0.97 * 0.01 + 0.25 * 0) = \frac{0.07275}{1.01},$$

  if the price goes down today.

  b) The price of the straddle today is approximately $\frac{0.23}{1.01}$. See activity a) of lesson 4.

Short-selling prohibition

- Activity 1

  a) The intersection between the cone of strategies (x_1, y_1) with $x_1 + 10y_1 \leq 0$ and $y_1 \geq 0$, and the cone of strategies with $1.02x_1 + 10.1y_1 \geq 0$ and $1.02x_1 + 9.7y_1 \geq 0$ is the point $(0, 0)$. This is not an arbitrage opportunity, and hence the model is arbitrage free.

  b) The cone of strategies (x_1, y_1) with $x_1 + 10y_1 \leq 0$ and $y_1 \geq 0$, and the cone of strategies with $(1+r)x_1 + 10.1y_1 \geq 0$ and $(1+r)x_1 + 9.7y_1 \geq 0$ meet in a point that represents and arbitrage opportunity when the slope of the line $(1+r)x_1 + 9.7y_1 = 0$ is greater than or equal to the one of the line $x_1 + 10y_1 = 0$ which happens if and only if $r \leq -0.03$.

- Activity 2

  If the stock pays dividens equal to $s\%$. The new low rate of return is $(1+d)(1+s) - 1 = 0.97(1+s) - 1$. In order to have $0.97(1+s) - 1 \geq r$, we need $s \geq \frac{1+r}{0.97} - 1$. In this case the model is no longer arbitrage free.

- We have that

$$\pi_0^{put} = \frac{1}{1.1} \sup_{0<q<1} (0 * q + 0.3(1-q)) = \frac{0.3}{1.1},$$

  and

$$\pi_0^{straddle} = \frac{1}{1.1} \sup_{0<q<1} (0.15 * q + 0.3 * (1-q)) = \frac{0.3}{1.1}.$$

Multi-factor models

- Activity 1

  The martingale measures in this case are given by triples (p, q, r) of positive numbers such that $p + q + r = 1$ and $1.65p + 1.5q + 1.2r = 1.5$. It is easy to verify that there exist infinitely many triples with such properties and hence the market is an arbitrage-free incomplete market. Consider a call option with maturity tomorrow and strike equal to \$1.5. To perfectly hedge this option an investor would have to solve the following system of equations

$$\begin{aligned} x_1 + 1.65y_1 &= 0.15 \\ x_1 + 1.5y_1 &= 0 \\ x_1 + 1.2y_1 &= 0. \end{aligned}$$

  This system does not have any solution.

- Activity 2

  In this case the risk neutral probabilities are quadruples of positive numbers (p, q, r, s) such that

$$\begin{aligned} p+q+r+s &= 1 \\ 1.01(p+q)+0.97(r+s) &= 1 \\ 1.02(p+r)+0.98(q+s) &= 1. \end{aligned}$$

  The system has infinitely many solutions and hence the market is arbitrage free but incomplete. Examples of solutions are $p = q = \frac{3}{8}$, $r = s = \frac{1}{8}$, or $p = \frac{5}{16}$, $q = \frac{7}{16}$, $r = \frac{3}{16}$, $s = \frac{1}{16}$.