Homework # 1 Math 3340 Spring 2021

Due at the end of the day 18 Feb 2021, at midnight.

Please submit your completed homework via gradescope on canvas. I encourage you to work with your classmates on this homework (except for the journal entries!) When you submit your work, please **list your collaborators**. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions **yourself**! You should include all computational details, and proofs should be carefully written with full details. As always, please write **neatly and legibly** (feel free to use LaTeX to write up your solutions, if you wish!).

Reading. *The Secret to Raising Smart Kids* by Carol Dweck, in Scientific American. You can find it by Google, or in the files section on our canvas site (it is also linked to from the syllabus).

Journal entry. Every now and then, I will ask you to write the equivalent of a journal entry. For this first journal entry, please include a picture of yourself (if possible), include the name you would prefer me to address you by, and also the time zone you are in. I'd really like to hear about why you are considering this course and what you hope to get out of it, any reactions to the Dweck article, and also a short "mathematical biography" of yourself. Finally, if there were one thing that you would like me to know about yourself, what would that be?

Exercises.

1. Suppose that $f : S \to T$ and $g : T \to U$ are both functions.

   (a) Show that if f and g are bijections, then $(gf)^{-1}$ exists and $(gf)^{-1} = f^{-1}g^{-1}$.

   (b) Prove or give a counter-example: If gf is injective, then f is injective.

   (c) Prove or give a counter-example: If gf is injective, then g is injective.

   (d) Prove or give a counter-example: If gf is surjective, then f is surjective.

   (e) Prove or give a counter-example: If gf is surjective, then g is surjective.

2. Let $S = \{1, 2\}$, and $T = \{3, 4, 5\}$.

   (a) How many functions $h : S \to T$ are there? How many are injective? How many are surjective? (As usual in this course, one must defend (with a reason or proof) your statement, and communicate to the reader why it is true. (i.e. merely giving an answer without explanation is worth very few points!).

   (b) How many functions $h : T \to S$ are there? How many are injective? How many are surjective?

3. Consider a square. Label the vertices 1, 2, 3, 4. Notice that every rotation or reflection of the square that maps every vertex to another vertex can be seen as a permutation in S_4. Find all such permutations, and write them in cycle notation. Find as small a set as possible that generates all the rest.

4. Consider the set of all permutations $S_n := \{f : [n] \to [n] \mid f \text{ is bijective}\}$.

   (a) Define $\sim$ on S_n by $\alpha \sim \beta$ if and only if there exists a $\sigma \in S_n$ such that $\beta = \sigma\alpha\sigma^{-1}$. Show that $\sim$ is an equivalence relation on S_n.

   (b) Suppose that α is the length k cycle $(1, 2, \ldots, k)$. If $\sigma \in S_n$, then show that $\sigma\alpha\sigma^{-1}$ is the k-cycle $(\sigma(1), \sigma(2), \ldots, \sigma(k))$.

   (c) Find $S_3/\sim$ (i.e. find all the equivalence classes). (Hint: use cycle notation).

   (d) Find $S_4/\sim$ (i.e. find all the equivalence classes).

   (e) Using your examples in the last two parts, formulate a proposition about this equivalence relation, or about $S_n/\sim$ (e.g. can you describe the equivalence classes, or, how many elements in an equivalence class, or, how many equivalence classes, etc?) This is an open ended problem. You do not need to prove your result, and indeed, it need not be correct, but it should be correct for $n = 3, 4$!

5. **Size of a set** Let's make the following definition: Given two sets X, Y (possibly infinite), X and Y have **the same size** if there exists a bijection $h \colon X \to Y$. We say that Y has **more elements (larger cardinality)** than X, if there does not exist a bijection between X and Y, but there is an injective map $h : X \to Y$.

   (a) Prove that 2 finite sets have the same size if and only if they have the same number of elements.

   (b) Determine whether $\mathbb{N}$ and $\mathbb{Z}$ have the same size. (Challenge question, not for homework: Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ have the same size).

   (c) Given a set S, let 2^S denote the set consisting of all subsets of S. Show that any function $f : S \to 2^S$ is not surjective by considering the set $\{s \in S \mid s \notin f(s)\}$ (and therefore 2^S has larger cardinality than S, even if S is infinite, and so, in this case, 2^S is a "larger" infinity).