Homework # 3 Math 3340 Spring 2021

Due at the end of the day 4 Mar 2021, at midnight.

Please submit your completed homework via gradescope on canvas. I encourage you to work with your classmates on this homework (except for the journal entries!) When you submit your work, please **list your collaborators**. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions **yourself**! You should include all computational details, and proofs should be carefully written with full details. As always, please write **neatly and legibly** (feel free to use LaTeX to write up your solutions, if you wish!).

Journal entry. There is no journal entry this week.

Exercises.

1. In this course, we accept (as an axiom) the well-ordering principle, stated below. We let $\mathbb{N}$ denote the set of all non-negative integers (natural numbers. But note, zero is an element of this set!).

   Well-ordering principle: Every non-empty subset of $\mathbb{N}$ contains a minimal element.

   In this problem, we assume this axiom (as we will for the entire course). Using this, we will prove that the method of proof by mathematical induction holds (i.e. the 2 following statements), and then apply it in an example.

   The principle of mathematical induction I: Let $P_1, P_2, \ldots,$ be a sequence of propositions (i.e. each P_k is either true, or false). Suppose that (1) P_1 is true, and (2) For all positive integers, if P_k is true, then P_{k+1} is true. Then P_m is true for all positive integers m.

   The principle of mathematical induction II: Let $P_1, P_2, \ldots,$ be a sequence of propositions. Suppose that (1) P_1 is true, and (2) For all positive integers k, if P_m is true for all $m \leq k$, then P_{k+1} is true. Then P_m is true for all positive integers m.

   (a) Prove the first principle of mathematical induction. You might want to start with "Let $T = \{n \in \mathbb{N} \mid n \geq 1 \text{ and } P_n \text{ is not true}\}$."

   (b) Prove the second principle of mathematical induction. Once again, you might want to start with "Let $T = \{n \in \mathbb{N} \mid n \geq 1 \text{ and } P_n \text{ is not true}\}$."

   (c) Prove using induction: For all positive integers n,

   $$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (The proposition P_n is that this formula holds for the given n).

2. (a) Let G be a group, and suppose $a, b \in G$ commute ($ab = ba$). Show that if $o(a)$ and $o(b)$ are finite and relatively prime, then $o(ab) = o(a)o(b)$.

(b) Let $\sigma \in S_n$ be a permutation, of cycle type $(n_1, n_2, \ldots, n_r)$, where $n_1 \geq n_2 \geq \ldots \geq n_r \geq 2$. (this means that the unique representation of σ as a product of disjoint cycles consists of cycles of length $n_1, n_2, \ldots, n_r$). For example $\sigma = (1,2,3)(4,5)(6,7,8)(9,10,11,12)$ has cycle type $(4,3,3,2)$. Show that the order of σ is the least common multiple of $\{n_1, \ldots, n_r\}$.

3. Let G be a group of order 6. We know from HW2 that G must contain an odd number of elements of order 2 (i.e., 1, 3, or 5).

   (a) Show that this number cannot be 5 (if it is, show that G would contain a subgroup of order 4).

   (b) If G is not cyclic, show that there exists an element $a \in G$ of order 2, and $b \in G$ of order 3, such that $ab \neq ba$.

   (c) If G is not cyclic, show that the multiplication table of S_3 and of G are the "same" (that is, one can name all the elements of each group, so that the multiplication tables are the same)

4. Let G be a group, and let $S \subseteq G$ be a subset. In this problem, we define and consider the subgroup $\langle S \rangle$ of G generated by S.

   (a) Show that the intersection of a (possibly infinite) collection of subgroups of G is also a subgroup.

   (b) Define
$$\langle S \rangle := \bigcap_{\{H \leq G | S \subseteq H\}} H$$
   to be the intersection of all subgroups of G containing S. Deduce that there is a unique minimal subgroup containing S. This is called the subgroup of G **generated by** S (or, the subgroup that S generates).

   (c) A word in S is an element of the form $g = w_1 w_2 \ldots w_r$, for any $r \geq 0$, where each w_i is in S, or $w_i^{-1} \in S$ (or both), and if $r = 0$, we mean $g = e$. Show that the group generated by S is precisely the set of words in S.

   (d) Show, using induction, that $S = \{(1,2), (2,3), \ldots, (n-1,n)\}$ generates S_n, for all $n \geq 2$.

5. In each of the following cases, determine if H is a subgroup of G (and prove your assertions).

   (a) Let $G \leq S_n$ be a subgroup.
$$H = \{\sigma \in G \mid \sigma(1) = 1\}.$$

   (b) Fix a positive integer $n > 2$. Consider the group $G = S_n \times S_n$.
$$H = \{(\sigma, \tau) \in G \mid \sigma(1) = \tau(1)\}.$$

   (c) Let G be a group, let
$$H = Z(G) = \{h \in G \mid gh = hg, \text{ for all } g \in G\}$$
   (called the center of G).

(d) Let G be a group, and let $a \in G$. Let

$$H = C_G(a) = \{g \in G \mid ga = ag\}$$

(H is called the centralizer of a in G).