Homework # 4 Math 3340 Spring 2021

Due at the end of the day Monday, 15 Mar, at midnight.

Please submit your completed homework via gradescope on canvas. I encourage you to work with your classmates on this homework (except for the journal entries!) When you submit your work, please **list your collaborators**. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions **yourself**! You should include all computational details, and proofs should be carefully written with full details. As always, please write **neatly and legibly** (feel free to use LaTeX to write up your solutions, if you wish!).

**Journal entry.** There is no journal entry this week.

**Exercises.**

1. Let $G$ be a finite group, and suppose that $H$ has index 2 in $G$. Show that $H$ is a normal subgroup.
2. Suppose that $G$ is a finite group. For $g \in G$, define $\rho_g : G \to G$ by $\rho_g(a) := ga$.
   (a) Show that for any $g$, $\rho_g$ is a bijection, and therefore an element of $Sym(G)$, or, in other words, is a permutation on the elements of $G$.
   (b) Show that for all $g, h \in G$, $\rho_{gh} = \rho_g \circ \rho_h$.
   (c) Define a function $\phi : G \to Sym(G)$ by setting $\phi(g) := \rho_g$.
   (d) Show that $\phi$ is a group homomorphism.
   (e) Find the kernel of $\phi$.
   (f) Prove that if $G$ has $n$ elements, then $G$ is isomorphic to a subgroup of $S_n$.
3. Let $G$ be a group. An automorphism of $G$ is an isomorphism $f : G \to G$. The set of all automorphisms of $G$ is denoted by $Aut(G)$.
   (a) Show that $Aut(G)$ is a group.
   (b) Show that for $g \in G$, the conjugation map $\tau_g : G \to G$, defined by $\tau_g(a) := gag^{-1}$ is an automorphism of $G$.
   (c) Define a function $\phi : G \to Aut(G)$ by $\phi(g) := \tau_g$. Show that $\phi$ is a homomorphism.
   (d) What is the kernel of $\phi$?
   (e) Find $Aut(\mathbb{Z}_3)$.
   (f) Find $Aut(S_3)$.
4. Find all of the subgroups of $D_4$. Determine which of these subgroups is normal.

5. In this problem, we consider even and odd permutations. A permutation $\sigma \in S_n$ is called **even** if it can be written as the product of an even number of transpositions, and it is called **odd** if it can be written as the product of an odd number of transpositions (so at the moment, it is possible that a permutation is both even and odd, but we prove in part (d) that this cannot happen). A **complete factorization** of $\sigma \in S_n$ is a representation of $\sigma$ as a product of disjoint cycles, where all 1-cycles are represented explicitly. For example, the complete factorization of $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 7 & 4 & 6 & 5 & 1 \end{bmatrix}$ is $(1\,3\,7)(2)(4)(5\,6)$. In this problem, you may use the fact that the complete factorization of a permutation $\sigma$ is unique, up to order.

   Define the **sign** of a permutation $\sigma \in S_n$ as follows: write the complete factorization of $\sigma$ to be: $\sigma = \sigma_1\sigma_2 \dots \sigma_m$. Define $\mathrm{sign}(\sigma) = (-1)^{n-m}$. For example, the sign of the above permutation is $(-1)^{7-4} = -1$. This permutation can be written as a product of 3 transpositions. But that doesn't (yet) make it odd, since perhaps there is another way of writing it as a product of an even number of transpositions. In this problem, we show that this cannot happen, and understand better what even and odd permutations are.

   (a) Given $k \geq 1$, what is the sign of a $k$-cycle in $S_n$?

   (b) Suppose that $k, \ell \geq 0$ and that $a, b, c_i, d_j$ are distinct integers in $[n]$. Show that

   $$(a\,b)(a\,c_1\,\dots\,c_k\,b\,d_1\,\dots\,d_\ell) = (a\,c_1\,\dots\,c_k)(b\,d_1\,\dots\,d_\ell)$$

   and

   $$(a\,b)(a\,c_1\,\dots\,c_k)(b\,d_1\,\dots\,d_\ell) = (a\,c_1\,\dots\,c_k\,b\,d_1\,\dots\,d_\ell)$$

   (c) Show that if $\tau$ is a transposition, and $\sigma \in S_n$, then $\mathrm{sign}(\tau\sigma) = -\mathrm{sign}(\sigma)$.

   (d) Show that if a permutation $\sigma \in S_n$ can be written as a product of an odd number of transpositions, then it can't be written as a product of an even number of transpositions. Conclude that every permutation is either even or odd, but not both.

   (e) Show that the function $\mathrm{sign} : S_n \longrightarrow \{1, -1\}$, where $\{1, -1\}$ is a group under multiplication, is a group homomorphism.

   (f) Let $A_n$ be the set of all even permutations in $S_n$. Show that it is a subgroup of $S_n$, and that this subgroup is normal.