Homework # 5 Math 3340 Spring 2021

Due at the end of the day Friday, Mar 26, at midnight.

Please submit your completed homework via gradescope on canvas. I encourage you to work with your classmates on this homework (except for the journal entries!) When you submit your work, please **list your collaborators**. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions **yourself**! You should include all computational details, and proofs should be carefully written with full details. As always, please write **neatly and legibly** (feel free to use LaTeX to write up your solutions, if you wish!).

Journal entry. There is no journal entry this week.

Exercises.

1. Let H and N be subgroups of a group G, where N is normal in G. Prove the following statements:
   (a) $HN \subset G$ is a subgroup. Recall that $HN = \{hn \mid h \in H, n \in N\}$.
   (b) $N \triangleleft HN$ is a normal subgroup.
   (c) Each element of HN/N has the form hN, for some $h \in H$.
   (d) $\phi\colon H \to HN/N$ defined by $\phi(h) = hN$ is a surjective homomorphism.
   (e) There is an isomorphism $H/H \cap N \to HN/N$.
   (f) Show that $|H||N| = |HN||H \cap N|$.
2. Show that if H and N are normal subgroups of a group G, $HN = G$, and $H \cap N = \{e\}$, then G is isomorphic to $H \times N$.
3. Suppose that G is a group of order pq, where p and q are primes. Suppose that G has a normal subgroup of order p, and also a normal subgroup of order q. Find G (that is, G is forced to be isomorphic to a group we "know". Find this group).
4. Suppose that $N \leq H \leq G$ are subgroups, and also suppose that H and N are normal subgroups in G. Consider $\phi\colon G/N \to G/H$ defined by $\phi(aN) = aH$.
   (a) Show that ϕ is well defined, is a homomorphism, and is surjective.
   (b) Show that $(G/N)/(H/N) \cong G/H$.
5. Consider the group $G = \mathbb{Z}_6 \times \mathbb{Z}_4$, let H be the cyclic subgroup generated by the element $(3, 2)$. Find G/H (i.e. find an isomorphism with a direct product of Abelian cyclic groups).
6. Let p be a prime number, and let G be a finite Abelian group such that p divides $|G|$. Show that G contains an element of order p. (Hint: use induction on the number of elements in the group G).

7. Prove that if G is an Abelian group of order 8, then G is isomorphic to one of the following groups:
   (a) $\mathbb{Z}_8$
   (b) $\mathbb{Z}_4 \times \mathbb{Z}_2$
   (c) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
8. Find all factor groups of the following groups:
   (a) $G = D_4$
   (b) $G = D_5$
   (c) $D = Q$, the group in problem 4, on the first prelim.