Math 3340
Spring 2021
Lecture #11

3 Mar 2021
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Today: homomorphisms + isomorphisms
kernel + image
normal subgroups

next:
Examples: $GL_n(\mathbb{F})$, D_n, Quaternions, ...

<u>Def</u> $f: G \longrightarrow H$ is a homom. of groups

if $f(ab) = f(a)f(b)$ $\forall\, a, b \in G$.

op. in G   op. in H

f is an <u>isom.</u> if it is a homom
+ it is bijective.

in this case: $G \cong H$ G is isomorphic to H.

Basic facts about homomorphisms

Prop Let $f: G \longrightarrow H$ be a homomorphism of groups. Then

(a) $f(e) = e$

(b) $f(a^{-1}) = f(a)^{-1} \quad \forall a \in G$

(c) $f(a^n) = f(a)^n \quad \forall n \in \mathbb{Z}$.

warning: if $(G,+)$ is written with $+$
$(H, \cdot)$ $\cdot$

$$f(na) = f(a)^n$$

$\boxed{\text{proof}}$ (a) $f(e \cdot e) = f(e) \cdot f(e) \overset{\text{cancellation}}{\Longrightarrow} e_H = f(e_G)$.
$\quad \| $
$f(e)$

(b) $f(a \cdot a^{-1}) = f(a) f(a^{-1})$
$\quad \|$
$f(e)$
$\quad \|$
e

also $\underbrace{f(a^{-1})}\, f(a) = e$

$\therefore f(a)^{-1} = f(a^{-1})$.

(c) $f(a^n) \overset{?}{=} f(a)^n$

$n = 0, 1, -1$: ⓐ ⓑ ✓.

prove for $n \geq 1$, $f(a^n) = f(a)^n$
by induction on n.

$n = 1$ ✓ trivial $f(a) = f(a)$ ✓

suppose true for $n-1$: assume $f(a^{n-1}) = f(a)^{n-1}$

show $f(a^n) = f(a)^n$:

$$f(a^n) = f(a \cdot a^{n-1})$$
$$= f(a)\, f(a^{n-1})$$
$$= f(a)\, f(a)^{n-1} \quad \text{induction hyp.}$$
$$= f(a)^n \quad ✓.$$

similarly: prove for $n < 0$, $f(a^n) = f(a)^n$.
leave this to you. //

$\boxed{\text{example}}$ Suppose G, H are both cyclic groups
both are order m. Are G and H isomorphism?
find $f : G \longrightarrow H$ an isomorphism.

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soln $G = \{e, a, a^2, \dots, a^{m-1}\}$ $a^m = e$.

for some $a \in G$.

$H = \{e, b, b^2, \dots, b^{m-1}\}$ $b^m = e$.

for some $b \in H$ (this $e \in H$)

$$f : G \longrightarrow H$$
$$e \longmapsto e$$
$$a \longmapsto b$$

if $f(a) = b$ and f is a homom.

then $f(a^2) = b^2, \dots, f(a^{m-1}) = b^{m-1}$

now f is defined as a function:

$a^i \longmapsto b^i$ $0 \le i \le m-1$.

to check f is a homom: (actually: for all i)

$$f(a^i a^j) \stackrel{?}{=} f(a^i) f(a^j)$$

(since every elem of G is of form a^i).

$$= f(a^{i+j}) = b^{i+j}$$

$$\shortparallel$$

$$f(a^i) f(a^j) = b^i b^j$$

$\therefore$ f is a homom.

f is bijective, $\therefore$ f is an isomorphism.

isomorphisms Suppose $f: G \longrightarrow H$ is an isomorphism

then
(a) If $a \in G$, $o(a) = o(f(a))$

(b) G is cyclic $\iff$ H is cyclic

(c) G is Abelian $\iff$ H is Abelian

to check if $\exists$ $f: G \longrightarrow H$ an isom:

(a) first: $|G| \neq |H| \implies$ NO ISOM

(b) if G is Abelian, H non Abelian $\implies$ NO ISOM

(c) all the orders of all the elements must match, otherwise: NO ISOM.

Example Consider groups of order 4.

$$V = \{e, (12)(34), (13)(24), (14)(23)\}$$

$$\mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_5^{\times}, \quad \mathbb{Z}_8^{\times}$$

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G = rotation of a square by 90°

6 of these,
which are isomorphic?

$\underline{\underline{\mathbb{Z}_4, G, \mathbb{Z}_5^\times}}$ these are all cyclic!
$\therefore$ these are all isomorphic

$V, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_8^\times$

V : 4 elements orders 1, 2, 2, 2

$\mathbb{Z}_2 \times \mathbb{Z}_2$: order 1, 2, 2, 2

$0 = [0]_2$
$1 = [1]_2$

$$\underset{e}{(0,0)}, \underbrace{(0,1), (1,0), (1,1)}_{\text{order } 2.}$$

V :

$a = (1\ 2)(3\ 4)$ $a^2 = e$

$b = (1\ 3)(2\ 4)$ $b^2 = e$

$ab = (1\ 4)(2\ 3)$ order 2.

"write": $V = \langle a, b \mid a^2 = e, b^2 = e, ab = ba \rangle$

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$\mathbb{Z}_2 \times \mathbb{Z}_2$: $a = (1,0)$ $2a = e$

$b = (0,1)$ $2b = e$

$a+b = (1,1)$

$$f : V \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$e \longmapsto e$$

$$a \longmapsto (1,0)$$

$$b \longmapsto (0,1)$$

$$ab \longmapsto (1,1)$$

f is bijection

$$f(gh) = f(g)f(h)$$

e.g.: $f(aa) = f(a) * f(a)$

$f(aa) = f(e) = (0,0)$, $f(a) * f(a) = (0,0)$.

check all of these.

$\therefore$ f is an isom.

$\mathbb{Z}_8^\times$ is also isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.