Math 3340
Spring 2021
Lecture #13

8 Mar 2021
Lecture #13

- NO CLASS Wednesday!

- office hours CHANGED this week
  (see announcement on canvas)

Burnside, 1905:

If $G \leq GL_n(\mathbb{C})$ of index d

(ie: every element $g \in G$ has order $ord(g) \leq d$)

then G is a finite group.

Rigid motions of $\mathbb{R}^2$ + dihedral groups

Rigid motion of $\mathbb{R}^2$:

is a bijective function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

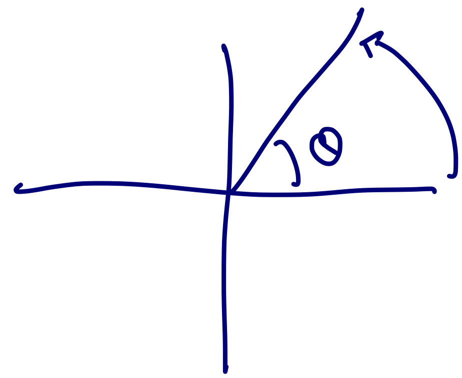
which preserves distances:

[ie: $\forall\ P, Q \in \mathbb{R}^2$, then

$$\| P - Q \| = \| f(P) - f(Q) \|$$]

3 special kinds:

① R_θ = rotation of angle θ ↺ CCW (if $\theta \geq 0$), rotation by $-\theta$ ↻ CW (if $\theta < 0$) about the origin



② Translation by $Q = (a,b)$

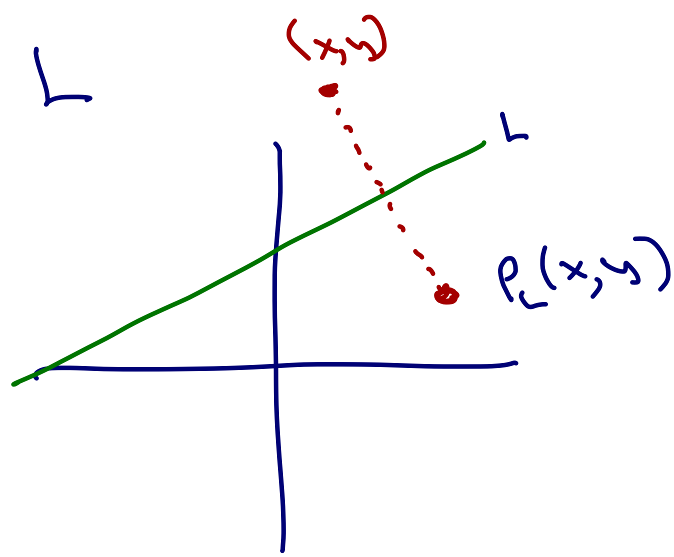
$$T_Q : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (x+a, y+b)$$

or $P \longmapsto P+Q$,

③ Reflection about a line L

$$\rho_L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$



We have the factoids about this

(1) M := set of all rigid motions, (rigid motions of $\mathbb{R}^2$)
w. composition of functions

= group.

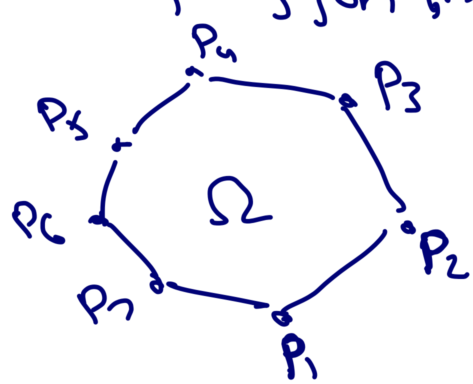
(2) M is generated by R_{θ}, ρ_L, T_Q $\forall \theta, L, Q$.

(3) if $P, Q \in \mathbb{R}^2$, $f \in M$

then $\overline{PQ}$ has $f(\overline{PQ}) = \overline{f(P)\, f(Q)}$

↑ line segment

if $P_1, \dots, P_n$ are consecutive vertices of a polygon ($n \geq 3$) Ω



$f(\Omega)$ = polygon with consecutive vertices $f(P_1), f(P_2), \dots, f(P_n)$.
(either CW or CCW about Ω.

also $f(\Omega)$ is congruent to Ω (same distances, angles as Ω)

(4) if $f \in M$, suppose P_1, P_2, P_3 are not on a common line.

if $f(P_i) = P_i \quad i = 1,2,3$

then $f = id_{\mathbb{R}^2} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Def If $\Omega \subseteq \mathbb{R}^2$ is a region (e.g: polygon) then $\boxed{\text{Symmetries}(\Omega)} := \{ f \in M : f(\Omega) = \Omega \}$

prop $G = \text{Symmetries}(\Omega)$ is a subgroup of M.

proof

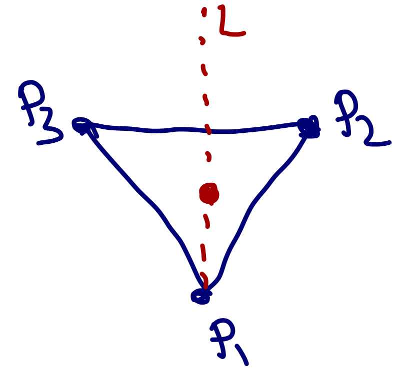
- $id_{\mathbb{R}^2} \overset{?}{\in} G$ ✓ yes!
- if $f, g \in G$, is $gf \in G$? yes ✓
- $f \in G \overset{?}{\implies} f^{-1} \in G$: $f(\Omega) = \Omega$

  apply f^{-1} : $f^{-1}f(\Omega) = f^{-1}(\Omega)$, where $f^{-1}f(\Omega) = \Omega$ //

$\therefore G \leq M$.

Example

Ω = equilateral triangle:



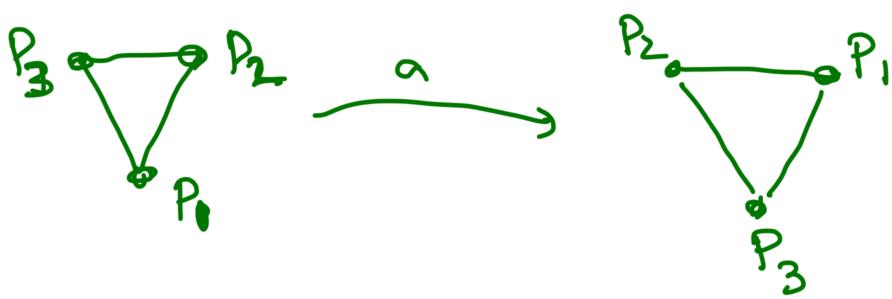
$$D_3 = \{ f \in M : f(\Omega) = \Omega \}$$

what is D_3?

$b = \rho_L$ reflection about y-axis

2 other reflections

$a = R_{120^\circ}$ (CCW) about center



$a^3 = e$

$b^2 = e$

e, a, a^2

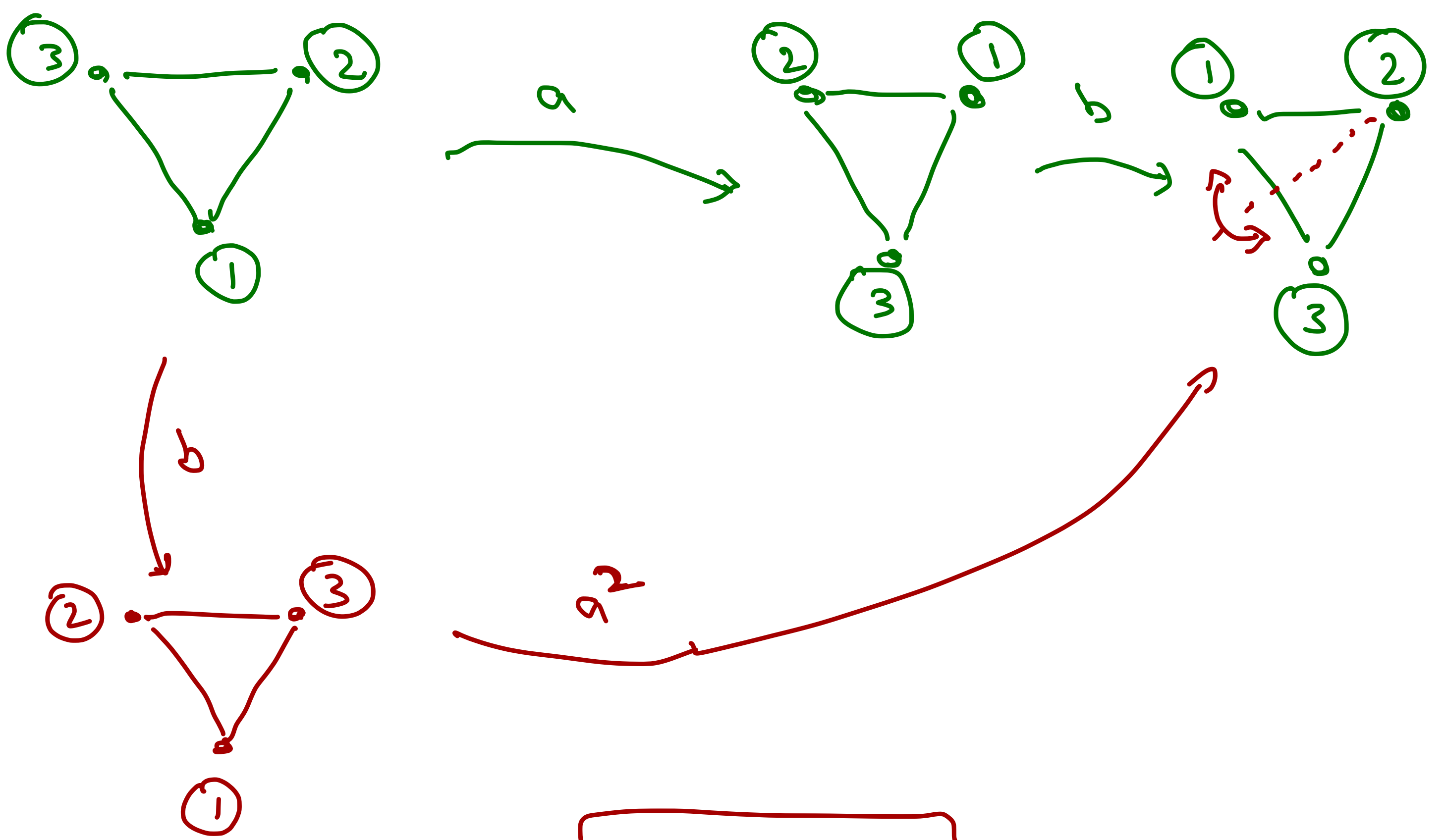
$b, ab, a^2 b$

any others? answer: no?!

each $f \in D_3$ permutes the vertices

if f fixes the 3 vertices then $f = e$.

$$ba \overset{?}{=} a^2 b$$



$\therefore$ $\boxed{ba = a^2 b}$

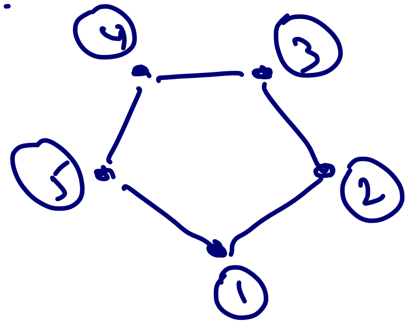
Sometimes write:

$$D_3 = \langle a, b \mid a^3 = e,\ b^2 = e,\ ba = a^2 b \rangle$$

"generators + relations view of a group".

In breakout:

(a) D_5 :

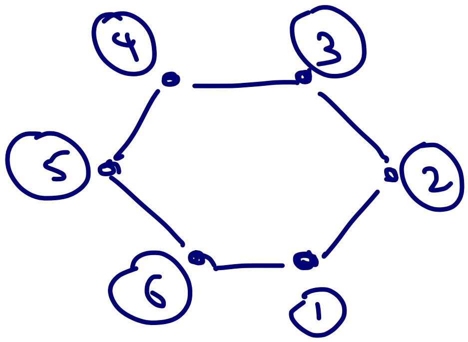


rows 4,5,6,7.

$|D_5| = 10$

find D_5

(b) D_6 :



rows 1,2,3

$|D_6| = 12$

$|D_5| = ?$ what are they.

$|D_6| = ?$

b = reflection about y-axis

a = rotation $R_{\frac{360}{5}} = R_{72}$ (D_5)

R_{60} (D_6)

theorem

(a) $|D_n| = 2n$

$f \in D_n$

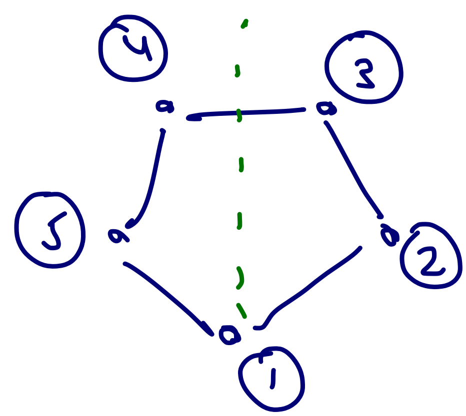
$f(P_1)$: n choices.

$f(P_2)$: adjacent or 2 choices

(b) D_n is isomorphic to a subgroup of S_n.

(c) $D_n = \langle a, b \mid a^n = e, b^2 = e, ba = a^{n-1} b \rangle$

D_5 as a subgroup of S_5



$a(P_1) = P_2$

$a = (1\ 2\ 3\ 4\ 5)$

$b = (2\ 5)(3\ 4)$

generated by these.