Homework # 3 Math 4310 Fall 2020

Due 9/23/20

Please submit your completed homework via gradescope on canvas. I encourage you to work with your classmates on this homework (except for the journal entries!) When you submit your work, please **list your collaborators**. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions **yourself**! You should include all computational details, and proofs should be carefully written with full details. As always, please write **neatly and legibly** (feel free to use LaTeX to write up your solutions, if you wish!).

Journal entry. There is no journal entry this week.

Exercises.

1. Find bases and the dimension of the following vector spaces (Make sure you believe that they are subspaces, but you do not need to prove that here!).
   (a) $W = \{A \in \mathbb{Q}^{2\times 2} \mid AB = BA\}$, where $B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$.
   (b) $U = \{\mathbf{v} \in (\mathbb{Z}_5)^4 \mid C\mathbf{v} = 0\}$, where
   $$C = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 4 & 2 & 4 & 1 \end{pmatrix}$$

2. (a) Consider $\mathcal{P}_2(\mathbb{R})$ of polynomials over $\mathbb{R}$ of degree at most 2. Find polynomials f_0, f_1, f_{-1} which satisfy the following conditions: $f_0(0) = 1$, $f_0(1) = f_0(-1) = 0$, $f_1(1) = 1, f_1(-1) = f_1(0) = 0$, and $f_{-1}(-1) = 1, f_{-1}(0) = 0, f_{-1}(1) = 0$. Is (f_0, f_1, f_{-1}) a basis of $\mathcal{P}_2(\mathbb{R})$? As always, prove your answer.
   (b) The Bernstein polynomials for degree n are $b_i = \binom{n}{i} x^i (1-x)^{n-i}$, for $i = 0, \ldots, n$. Note that we always assume that the binomial coefficient $\binom{a}{0} = 1$, for $a > 0$. For example, for $n = 2$, we have $(1-x)^2, 2x(1-x), x^2$. Prove that these three polynomials form a basis of $\mathcal{P}_2(\mathbb{R})$. Challenge (definitely optional!): prove that the $n+1$ Bernstein polynomials of degree n form a basis of $\mathcal{P}_n(\mathbb{R})$.

3. A permutation on $[n] := \{1, 2, \ldots, n\}$ is a (re)-arrangement of these numbers. For example, $(2, 3, 1, 4)$ is a permutation on $[4]$, as is $(1, 2, 3, 4)$.

   An $n \times n$ permutation matrix is an $n \times n$ matrix which has exactly one "1" in each row and in each column, and all other entries are zero.

   (a) Find all 3×3 permutation matrices over $\mathbb{R}$. Let $V \subset \mathbb{R}^{3\times 3}$ be the subspace spanned by these matrices. Find a basis for this subspace. What is its dimension? (To explore, but not for homework: what happens for $n = 4$, or higher n?)

(b) Show that there is a 1-1 correspondence between permutations on $[n]$ and $n \times n$ permutation matrices. Show that there are $n!$ $n \times n$ permutation matrices.

4. For this problem (and for any problems or exams later in the course), you may use the following integrals. If m and n are positive integers, then

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0$$

Consider the subspace $V_r \subset \mathcal{F}un(\mathbb{R}, \mathbb{R})$ spanned by the functions

$$V_r := span(1, \sin(x), \sin(2x), \ldots, \sin(rx), \cos(x), \cos(2x), \ldots, \cos(rx))$$

(a) Show that $(1, \sin(x), \cos(x))$ is linearly independent, and therefore a basis of V_1.

(b) Is $(1, \sin^2(x), \cos^2(x))$ linearly independent? Prove or disprove.

(c) Show that $(1, \sin(x), \cos(x), \sin(2x), \cos(2x))$ is linearly independent, and therefore a basis of V_2. (Hint: remember that if you have two functions that are equal to each other, then multiplying both sides by the same function, integrating both (using definite integral) sides, differentiating both sides, evaluating both sides at the same value, gives two things which are still equal).

5. In this problem, you may assume that the space of infinite sequences in $\mathbb{F}$,

$$\mathbb{F}^\infty = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{F}, \text{for all } i > 0\}$$

is also a vector space. (Note: there is no convergence conditions on these sequences!)

Define a subset

$$W = \{(a_1, a_2, a_3, \ldots) \mid a_i = a_{i-1} + a_{i-2}, \text{for } i \geq 3\}.$$

You may also assume that W is a subspace, although you should check for yourself that it is one!

(a) Is W finitely generated? If so, find a basis for W. What is its dimension?

(b) Find a complementary space U of W, i.e. a subspace U such that $U \bigoplus W = \mathbb{F}^\infty$.

6. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U, W, X be subspaces.

(a) If $U \cap W = 0$, prove that $\dim(U + W) = \dim U + \dim W$.

(b) In general, prove $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.

(c) If you're familiar with the "inclusion-exclusion principle", you might guess that part (b) generalizes to the equality

$$\dim(U+W+X) = \dim(U)+\dim(W)+\dim(X)-\dim(U\cap W)-\dim(U\cap X)-\dim(W\cap X)+\dim(U\cap W\cap X).$$

Provide a counterexample to show this "equality" can be false!
(Hint: You can take the field $\mathbb{F}$ to be $\mathbb{R}$ and the vector space V to be $\mathbb{R}^2$).