Revised Homework # 4 Math 4310 Fall 2020

Due 9/30/20

Please submit your completed homework via gradescope on canvas. I encourage you to work with your classmates on this homework (except for the journal entries!) When you submit your work, please **list your collaborators**. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions **yourself**! You should include all computational details, and proofs should be carefully written with full details. As always, please write **neatly and legibly** (feel free to use LaTeX to write up your solutions, if you wish!).

Journal entry. There is no journal entry this week.

Exercises.

1. (a) If $\mathbb{F} = \mathbb{Z}_p$, where p is a prime number, and V is a vector space of dimension n over $\mathbb{F}$, how many elements does V have?
   (b) How many (ordered) bases do the vector spaces $\mathbb{F}^1$ and $\mathbb{F}^2$ have, if $\mathbb{F} = \mathbb{Z}_2$? if $\mathbb{F} = \mathbb{Z}_p$?.
   (c) How many (ordered) bases does the vector space $\mathbb{F}^3$ have, if $\mathbb{F} = \mathbb{Z}_2$? if $\mathbb{F} = \mathbb{Z}_p$?.

2. Given a vector space V over a field $\mathbb{F}$, we define the **dual vector space** $V^* := \mathcal{L}(V, \mathbb{F})$. If V is finite dimensional of dimension n, find the dimension of V^*, and find a basis of this vector space.

3. Find the dimensions of the following vector spaces. Suppose that $\dim U = a$, $\dim V = b$, and $\dim W = c$. (Recall that for this course, you must always give reasons for your answers, simply writing down the dimension is not good enough!) You may use the fact that $\dim \mathcal{L}(V, W) = \dim V \dim W$ without proof in this problem. All sums here are direct products (aka external direct sums), not our original notion of direct sum!
   (a) $U \times V$ (note: we often write this as $U \oplus V$, even though U and V might not be subspaces of a common vector space. We do this below too).
   (b) $\mathcal{L}(V, U)$.
   (c) $\mathcal{L}(V, U \oplus U)$.
   (d) $U \oplus V \oplus W$.
   (e) $\mathcal{F}un([2] \times [2] \times [2], \mathbb{F})$, where $[n] = \{1, 2, \ldots, n\}$.

4. Find bases for both the kernel and image of the following maps, together with their dimensions. Which is injective, which is surjective? You may assume that these are all linear maps.
   (a) $T : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R})$, where $T(f(x)) := xf(x) + f'(x)$.
   (b) $T : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ defined by $T(A) := AB - BA$, where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(c) $D : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$, defined by $D(f(x)) = f'(x)$.

5. Let V be a finite dimensional vector space over a field $\mathbb{F}$, let X be a non-empty finite set, and let $\mathcal{F}un(X, V)$ denote the vector space of functions from the set X to the vector space V. (You do not need to prove that this is a vector space over $\mathbb{F}$.) Fix an element $a \in X$, and define a function

$$\begin{aligned} T_a : \mathcal{F}un(X, V) &\rightarrow V \\ f &\mapsto f(a). \end{aligned}$$

Prove that T_a is a linear transformation, and find $\ker(T_a)$ and $\mathrm{Im}(T_a)$. What are the dimensions of $\mathcal{F}un(X, V)$, $\ker(T_a)$ and $\mathrm{Im}(T_a)$? Finding $\ker(T_a)$ does not mean to necessarily find a basis. You can also describe it as all functions that satisfy certain properties. Of course, you can use bases too, if you show they are bases.

6. Let $T : \mathbb{F} \longrightarrow \mathbb{F}$ be defined to be $T(x) = x^3$.

(a) Is this a linear map if $\mathbb{F} = \mathbb{Z}_2$? (Either prove, or show why not).

(b) Is this a linear map if $\mathbb{F} = \mathbb{Z}_3$? (Either prove, or show why not).

(c) Is this a linear map if $\mathbb{F}$ has characteristic not equal to 3? (Either prove, or show why not).

7. Recall that if $z = a + bi \in \mathbb{C}$, where a, b are real numbers, then its conjugate is $\bar{z} := a - bi$. Consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$, where $f(z) = \bar{z}$.

(a) If we consider this a map of $\mathbb{C}$-vector spaces, is f a linear map?

(b) If we consider this a map of $\mathbb{R}$-vector spaces, is f a linear map?