Homework # 6 and #7 (revised)

Math 4310 Fall 2020

Due Friday, 10/30/20

Please submit your completed homework via gradescope on canvas. I encourage you to work with your classmates on this homework (except for the journal entries!) When you submit your work, please **list your collaborators**. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions **yourself**! You should include all computational details, and proofs should be carefully written with full details. As always, please write **neatly and legibly** (feel free to use LaTeX to write up your solutions, if you wish!).

Journal entry. There is no journal entry this week.

Exercises.

The first 4 problems form HW6 (these are about quotient vector spaces), and the last 4 problems form HW7 (these are about dual vector spaces and dual linear maps).

1. Suppose that $T : V \longrightarrow W$ is a linear map (V and W are vector spaces, not necessarily finite dimensional). Define a new map $\widetilde{T} : V/\ker T \longrightarrow W$, defined as $\widetilde{T}(\mathbf{v} + \ker T) = T(\mathbf{v})$.

   (a) Prove that $\widetilde{T}$ is well defined.

   (b) Prove that $\widetilde{T}$ is a linear map.

   (c) Prive that $\ker \widetilde{T} = 0$.

   (d) Prove that if T is surjective, then $\widetilde{T}$ is an isomorphism. (This is a good way to get isomorphisms of quotient vector spaces with something more "friendly"!).

   (e) Suppose that $V = U \oplus W$, where U and W are subspaces of V. Consider the linear map $T : V \longrightarrow W$, which sends $\mathbf{u} + \mathbf{w}$ to $T(\mathbf{u} + \mathbf{w}) = \mathbf{w}$. What can you say about $\widetilde{T}$ for this example? (You may use in later problems what you learn in this problem!)

2. Let V be a vector space over the field $\mathbb{F}$, and let W be a subspace. Recall that a *complement* of W is a vector subspace of V such that $V = U \oplus W$ (and we call U and W complementary).

   If U_1 and U_2 are both complements of W in V:

   (a) Show that U_1 and U_2 are isomorphic.

   (b) Is $U_1 = U_2$? Either prove this, or give a counter-example.

   (c) Let $O(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = -f(x)\}$ be the set of **odd** smooth functions and let $E(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = f(x)\}$ be the set of **even** smooth functions. Prove that $O(\mathbb{R})$ and $E(\mathbb{R})$ are complementary. Prove that $C^\infty(\mathbb{R})/O(\mathbb{R}) \cong E(\mathbb{R})$.

3. Let U and V be vector spaces with respective subspaces X and Y. Prove that there is an isomorphism $(U \times V)/(X \times Y) \cong (U/X) \times (V/Y)$.

4. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f : \mathbb{R} \to \mathbb{R}$. (Note: $C^\infty(\mathbb{R})$ is **very** infinite-dimensional!)

   Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of those functions which "vanish to n^{th} order at 0":
$$W = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; f(0) = 0,\ \frac{df}{dx}(0) = 0, \ldots, \text{ and } \frac{d^n f}{dx^n}(0) = 0 \right\}.$$
   Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite-dimensional and find a basis.

5. Suppose that m is a positive integer, and let $V = \mathcal{P}_m(\mathbb{R})$ be the vector space of polynomials of degree at most m. Consider the basis $\mathcal{A} = (1, x, x^2, \ldots, x^m)$ of V.

   (a) Show that the dual basis to $\mathcal{A}$ is $\mathcal{A}^* = (\phi_0, \phi_1, \ldots, \phi_m)$, where
$$\phi_j(p(x)) = \frac{p^{(j)}(0)}{j!}.$$

   (b) Show that $\mathcal{B} = (1, x-3, (x-3)^2, \ldots, (x-3)^m)$ is a basis of V.

   (c) Find the dual basis $\mathcal{B}^*$.

6. Suppose that $\mathcal{A} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis for a vector space V, and that $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ is a basis for a vector space W. Suppose that $T : V \longrightarrow W$ is a linear map. Also suppose that the dual bases of V^* and W^* are (respectively) $\mathcal{A}^*$, $\mathcal{B}^*$. If $A = (A_{i,j})$ is the matrix $[T]_{\mathcal{B} \leftarrow \mathcal{A}}$, then show that the matrix of $T^* : W^* \longrightarrow V^*$ (i.e. $[T^*]_{\mathcal{A}^* \leftarrow \mathcal{B}^*}$), is the transpose of A.

7. Assume that V is a finite dimensional vector space. Recall that if $U \subset V$ is a subspace of V, then
$$U^o := \{\varphi \in V^* \mid \varphi(u) = 0 \text{ for all } u \in U\}.$$

   (a) The inclusion map $i : U \longrightarrow V$ is the linear map sending $\mathbf{u} \in U$ to $\mathbf{u}$ (but in V). By using the fundamental theorem of linear algebra on the dual of this map, show that $\dim U + \dim U^o = \dim V$.

   (b) For which $U \subseteq V$ is $U^o = 0$? For which U is $U^o = V^*$? (State a result, and prove it).

8. Assume that V and W are finite dimensional vector spaces. Let $T : V \longrightarrow W$ be a linear map with dual $T^* : W^* \longrightarrow V^*$. (In class, we used the notation T^t, but I like the notation T^* better). Recall that $T^*(\varphi) = \varphi T$, where $\varphi : W \longrightarrow \mathbb{F}$.

   (a) Show that $\ker(T^*) = (\mathrm{image}(T))^o$.

   (b) Show that $\dim \mathrm{image}(T) = \dim \mathrm{image}(T^*)$

   (c) Show that $\mathrm{image}(T^*) \subset (\ker(T))^o$. Show that $\mathrm{image}(T^*)$ and $(\ker(T))^o$ have the same dimension, and therefore that these subspaces of V^* are the same.

   (d) Show that T^* is surjective if and only if T is injective.

   (e) Show that T^* is injective if and only if T is surjective.

9. This problem is **being moved** to the **next** homework set! I include it here, but it is not due this week!. Suppose that $A \in \mathbb{F}^{m \times n}$, and let $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be the corresponding linear map. We often represent elements of $\mathbb{F}^n$ as column vectors. Elements of $\mathbb{F}^{n*} = \mathcal{L}(\mathbb{F}^n, \mathbb{F}^1)$ can be thought of as $1 \times n$ matrices, i.e. row vectors.

   Previous description, not as clear: Similarly, we often represent elements of $\mathbb{F}^{n*}$ as row vectors, where the row vector e_i^* (with a 1 in the ith column, zeros elsewhere, corresponds to $\varphi_i \in V^*$.

   A matrix A has 4 interesting subspaces associated with it:

   (i) $\mathit{colspan}(A) \subset \mathbb{F}^m$ is the span of the columns of A,

   (ii) $\mathit{rowspan}(A) \subset \mathbb{F}^{n*}$ is the span of the rows of A,

   (iii) $\ker(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$,

   (iv) the left kernel of A, $\mathit{leftker}(A) = \{\mathbf{y} \in \mathbb{F}^{m*} \mid \mathbf{y}A = \mathbf{0}\}$.

   For this problem, suppose that $T = L_A$.

   (a) If $\mathbf{w} = (a_1, \ldots, a_n)$ is a row vector (i.e. a $n \times 1$ matrix) representing an element of $\mathbb{F}^{n*} = \mathcal{L}(\mathbb{F}^n, \mathbb{F}^1)$, and if $\mathbf{v} = b_1 e_1 + \ldots + b_n e_n \in \mathbb{F}^n$ is a column vector, what is $\varphi(\mathbf{v})$?

   (b) In the previous problem, several subspaces are mentioned:

   $$\ker(T^*), \ker(T), \mathit{image}(T), \mathit{image}(T^*), (\ker(T))^o, (\mathit{image}(T))^o.$$

   Some of these are the same as the 4 subspaces associated to A above. Match them up, and prove that they are the same.