Lecture #4 4310 Fall 202

9 Sep 2020
Lecture 4

①

HW#1 due today

last time:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, \ldots, a_n \in \mathbb{F} \right\}.$$

defined $\vec{v} + \vec{w}$, $c\vec{v}$ $c \in \mathbb{F}$, $\vec{v}, \vec{w} \in \mathbb{F}^n$

Def of a vector space

Let $\mathbb{F}$ be a field, and let V be a set equipped with 2 operations

$\alpha : V \times V \longrightarrow V$ "addition"

write $\alpha(\vec{u}, \vec{v}) = \vec{u} + \vec{v}$.

$\mu : \mathbb{F} \times V \longrightarrow V$ "scalar multiplication"

write $\mu(c, \vec{v}) = c\vec{v}$ or $c \cdot \vec{v}$

and one special element

$\vec{0}$ ($= \vec{0}_V$ sometimes) $\in V$.

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(2)

$(V, +, \text{scalar mult}, 0)$

is a vector space over $\mathbb{F}$ if

VS1 comm. of $+$: $\forall\, \vec{u}, \vec{v} \in V$, $\boxed{\vec{u} + \vec{v} = \vec{v} + \vec{u}}$

VS2 associativity of $+$, smult
$(\forall\, \vec{u}, \vec{v}, \vec{w} \in V)$
$a, b \in \mathbb{F}$

$\boxed{\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}}$

$\boxed{a \cdot (b\vec{v}) = (ab) \cdot \vec{v}}$

VS3 additive identity $\boxed{\vec{v} + \vec{0} = \vec{v}}$

VS4 additive inverse
$\forall\, \vec{v} \in V$, $\exists\, \vec{w} \in V$ s.t. $\boxed{\vec{v} + \vec{w} = \vec{0}}$
(later: this $\vec{w}$ is unique, written $-\vec{v}$).

VS5 mult. identity
$\forall\, \vec{v} \in V$

$\boxed{1_{\mathbb{F}} \cdot \vec{v} = \vec{v}}$

VS6 distributivity
$\forall\, a, b \in \mathbb{F}$
$\vec{u}, \vec{v} \in V$

$a \cdot (\vec{u} + \vec{v}) = (a \cdot \vec{u}) + (a \cdot \vec{v})$

$(a + b) \cdot \vec{u} = (a \cdot \vec{u}) + (b \cdot \vec{u})$

Examples

1. $\mathbb{F}^n$ is a vector space over $\mathbb{F}$

(with $+$, smult we defined last time).

$$0_{\mathbb{F}^n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (n \text{ entries}).$$

(this is a theorem. See the book for a proof),

easy to check VS1 – VS6 here.

2. (breakout room).

Define $\mathbb{F}^\infty = \{ (a_1, a_2, a_3, \ldots) \ : \ \text{each } a_i \in \mathbb{F} \}$.

Find what $+$, scalar mult, 0 element should be, to make $\mathbb{F}^\infty$ a vector space.

$\vec{0} = (0, 0, 0, \ldots) \in \mathbb{F}^\infty$.

$\vec{a} = (a_1, a_2, a_3, \ldots)$

$\vec{b} = (b_1, b_2, b_3, \ldots)$

$$\vec{a}+\vec{b} = (a_1+b_1, a_2+b_2, \ldots)$$

$$c\,\vec{a} = (ca_1, ca_2, ca_3, \ldots)$$

<u>Prop.</u> $\mathbb{F}^\infty$, with these operations, is a vector space over $\mathbb{F}$.

(3) $m \times n$ matrices over $\mathbb{F}$

$\mathbb{F}^{m\times n}$ = set of all $m\times n$ matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{m1} & & \cdots & A_{mn} \end{pmatrix}$$

with each $A_{ij} \in \mathbb{F}$. $\forall\ i,j$:
$1 \le i \le m$
$1 \le j \le n$

to make this a vector space

$A+B$ usual matrix addition.

cA usual scalar mult

$$= \begin{pmatrix} cA_{11} & \cdots & cA_{1n} \\ \vdots & & \\ cA_{m1} & \cdots & cA_{mn} \end{pmatrix}$$

0 element
$\parallel$ this is the
$m \times n$ zero matrix.

Prop $\mathbb{F}^{m \times n}$ is a vector space over $\mathbb{F}$.

($\mathbb{R}^{3\times 2}$ 3×2 matrices over $\mathbb{R}$)
($\mathbb{C}^{3\times 2}$ " $\mathbb{C}$)
($\mathbb{F}_4^{3\times 2}$ " over $\mathbb{F}_4$) (4^6 elements in it)

④ $\mathrm{Fun}(X, \mathbb{F})$ $\mathbb{F}$ = field

Let X be a set ① e.g: $[n] := \{1, 2, \ldots, n\}$
(warning: some use $[n] := \{0, 1, \ldots, n-1\}$).

② $\mathbb{R}$ (even if $\mathbb{F} = \mathbb{Z}_2$)

Let $\mathrm{Fun}(X, \mathbb{F}) = \{ f : X \to \mathbb{F} : f \text{ is a function} \}$

(if $a \in X$, $f(a) \in \mathbb{F}$ always some value!)

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Let's make $\mathrm{Fun}(X, \mathbb{F})$

a vector space over $\mathbb{F}$.

need to define: 0 element

$f+g$

cf

$c \in \mathbb{F}$

$f, g \in \mathrm{Fun}(X, \mathbb{F})$.

0 : $0_{\mathrm{Fun}(X,\mathbb{F})}(a) = 0_{\mathbb{F}}$.

$a \in X$

ie: $0 : X \longrightarrow \mathbb{F}$

$a \longmapsto 0$.

$f+g$: $f+g : X \longrightarrow \mathbb{F}$

$a \longmapsto f(a) + g(a)$

$+$ in $\mathbb{F}$

$cf : X \longrightarrow \mathbb{F}$

$a \longmapsto cf(a)$

mult in $\mathbb{F}$.

Prop $\mathrm{Fun}(X, \mathbb{F})$ with these definitions is a vector space over $\mathbb{F}$.
(try this on your own).