Lecture #5

Next up:
- Subspaces
- Sums
- Polynomials

11 Sep 2020
Lecture 5

HW #2 has been posted

office hours on Monday: for Stillman

new times: prob. be 3:45 – 4:45 pm

9 – 10 pm

Today: subspaces

example

e.g. is the subset $U = \left\{ \begin{pmatrix} a \\ b+a \\ b-a \end{pmatrix} : a, b \in \mathbb{R} \right\}$

$\subseteq \mathbb{R}^3$

is it a vector space?

ie: with the induced operations (A) we mean this.

or with some strange ops +, smult

Def Let V be a vector space over $\mathbb{F}$

A subset $U \subseteq V$ is called a subspace (of V)

is U is itself a vector space, under

the induced operations of +, smult

from V.

9 Sep 2020
Lecture 4 ②
(copied to Lecture 5)

$(V, +, \text{scalar mult}, 0)$

is a vector space over $\mathbb{F}$ if

✓ VS1 comm. of + : $\forall \vec{u}, \vec{v} \in V$, $\boxed{\vec{u} + \vec{v} = \vec{v} + \vec{u}}$

✓ VS2 associativity of +, smult $(\forall \vec{u}, \vec{v}, \vec{w} \in V)$, $a, b \in \mathbb{F}$

$\boxed{\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}}$

$\boxed{a \cdot (b\vec{v}) = (ab) \cdot \vec{v}}$

✓ VS3 additive identity $\boxed{\vec{v} + \vec{0} = \vec{v}}$

VS4 additive inverse
?
$\forall \vec{v} \in V$, $\exists \vec{w} \in V$ st. $\boxed{\vec{v} + \vec{w} = \vec{0}}$

(later: this $\vec{w}$ is unique, written $-\vec{v}$).

✓ VS5 mult. identity $\forall \vec{v} \in V$ $\boxed{1_{\mathbb{F}} \cdot \vec{v} = \vec{v}}$

✓ VS6 distributivity $\forall a, b \in \mathbb{F}$, $\vec{u}, \vec{v} \in V$

$\boxed{a \cdot (\vec{u} + \vec{v}) = (a \cdot \vec{u}) + (a \cdot \vec{v})}$

$\boxed{(a+b) \cdot \vec{u} = (a \cdot \vec{u}) + (b \cdot \vec{u})}$

$\boxed{\text{theorem}}$

Let V be a vector space over $\mathbb{F}$

$\mathcal{U}$ a subset of V.

Then

$\mathcal{U}$ is a subspace of V

$\iff$ (a) $0_V \in \mathcal{U}$

(b) $\mathcal{U}$ is closed under addition:

$\vec{v}, \vec{w} \in \mathcal{U}, \Rightarrow \vec{v}+\vec{w} \in \mathcal{U}$

(c) $\mathcal{U}$ is closed under scalar mult:

$c \in \mathbb{F}$, $\vec{v} \in \mathcal{U}$, then $c\vec{v} \in \mathcal{U}$.

proof $\Rightarrow$ (a), (b) and (c) hold

since $\mathcal{U}$ is a vector space.

$\Leftarrow$ suppose (a), (b), (c) hold.

show that $\mathcal{U}$ is a vector space

show VS1 – VS6 all but VS4 are immediate

VS4 : use $(-1)\cdot\vec{v} = (-\vec{v})$ cm fact on HW#2.

$\forall \vec{v} \in V$.

$\therefore -\vec{v} \in \mathcal{U}$ if $\vec{v} \in \mathcal{U}$. //

Examples

1. $V = \mathbb{F}^3$ Is U a subspace

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : \begin{array}{c} a+b+c=0 \\ a,b,c \in \mathbb{F} \end{array} \right\} \subseteq \mathbb{F}^3.$$

yes. ? $\in U$.

$0_V = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in U$ ✓

check b, c

check $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{pmatrix}$

$\in U$ $\in U$

$a_1+b_1+c_1=0$ $a_2+b_2+c_2=0$

similarly smult.

$\therefore$ U is a subspace of $\mathbb{F}^3$.

2. $V = \mathrm{Fun}(\mathbb{R}, \mathbb{R})$

$C(\mathbb{R},\mathbb{R}) = \{ f \in V : f \text{ is continuous} \}$

$C^\infty(\mathbb{R}) = \{ f \in V : f \text{ is infinitely diff.} \}$

by calculus, these are closed under $+$, smult.
(and 0 is in there too)

$\therefore$ they are subspaces.

Sums and direct sums

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Lecture 5

Let $U_1, U_2, \ldots, U_m \subseteq V$ be subspaces

<u>Def</u> $U_1 + U_2 + \cdots + U_m$

$$= \{ \vec{u}_1 + \vec{u}_2 + \cdots + \vec{u}_m : \vec{u}_i \in U_i \text{ for all } i \}$$

$$\subseteq V$$

<u>example</u> U_1 = xy plane in $\mathbb{R}^3$

$$= \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

U_2 = z axis in $\mathbb{R}^3$ $= \left\{ \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$

U_3 = xz plane. in $\mathbb{R}^3$

U_4 = ($x=y$ line in $z=0$)

$$= \left\{ \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$$

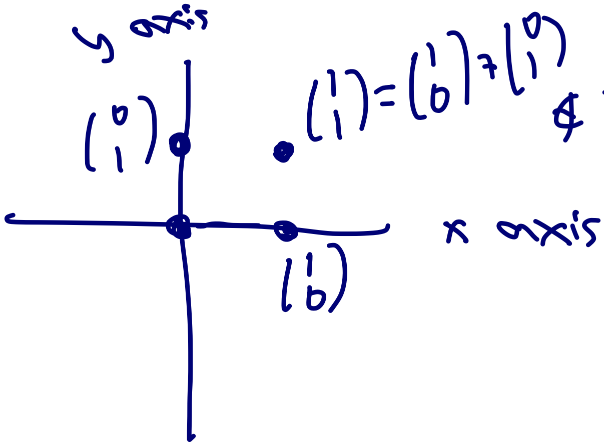
(a) $U_1 + U_2 = \mathbb{R}^3$

(b) $U_2 + U_3 = U_3$ xz plane

(c) $U_3 + U_4 = \mathbb{R}^3$

$(x\text{-axis}) \cup (y\text{-axis}) \overset{?}{=}$ subspace

NO



$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \notin$ union (in $\mathbb{R}^2$)

<u>prop</u> $U_1 + \cdots + U_m \subseteq V$ is also a subspace (if $U_1, \dots, U_m$ are subspaces).

<u>def</u> $U := U_1 + \cdots + U_m \subseteq V$ is called a <u>direct sum</u> if whenever

$\vec{u} = \vec{u}_1 + \vec{u}_2 + \cdots + \vec{u}_m \qquad (\vec{u}_i \in U_i)$

then there is exactly one way to write $\vec{u}$ like this.

ugh, what a junky name!

we write $U = U_1 \oplus U_2 \oplus \cdots \oplus U_m$

in this case (<u>if</u> sum is direct)