Lecture #6

14 Sep 2020
Lecture #6 ①

Recall:

$U = U_1 + \cdots + U_m = \{\vec{u}_1 + \cdots + \vec{u}_m : \vec{u}_i \in U_i \text{ all } i\} \subseteq V$

the sum is $\boxed{\text{direct}}$ if whenever

$\vec{u}_1 + \cdots + \vec{u}_m = \vec{w}_1 + \cdots + \vec{w}_m \quad (\vec{u}_i, \vec{w}_i \in U_i)$

then $\vec{u}_i = \vec{w}_i$ for all i,

(then write $U = U_1 \oplus \cdots \oplus U_m \subseteq V$)

while you wait

prove: $U_1 + \cdots + U_m$ is a direct sum

$\iff$ if $\vec{0} = \vec{u}_1 + \cdots + \vec{u}_m$, $\vec{u}_i \in U_i$ RHS

then $\vec{u}_1 = \cdots = \vec{u}_m = \vec{0}$.

proof $\Rightarrow$ assume $U_1 + \cdots + U_m$ direct sum

if $\vec{0} = \vec{u}_1 + \cdots + \vec{u}_m$ $\quad \vec{u}_i \in U_i$

show: $\vec{u}_1 = \cdots = \vec{u}_m = 0$

implies $\vec{0} = \vec{0} + \cdots + \vec{0} = \vec{u}_1 + \cdots + \vec{u}_m$

def $\Rightarrow$ $\vec{0} = \vec{u}_1, \vec{0} = \vec{u}_2, \ldots, \vec{0} = \vec{u}_m$.

$\Leftarrow$ want to show: $\vec{u}_1 + \cdots + \vec{u}_m = \vec{w}_1 + \cdots + \vec{w}_m$

then $\vec{u}_1 = \vec{w}_1, \ldots, \vec{u}_m = \vec{w}_m$

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(2)

subtract:

$(\vec{u}_1 - \vec{w}_1) + (\vec{u}_2 - \vec{w}_2) + \dots + (\vec{u}_m - \vec{w}_m) = \vec{0}$.

RHS $\Rightarrow$ $\vec{u}_1 - \vec{w}_1 = \dots = \vec{u}_m - \vec{w}_m = \vec{0}$

$\therefore \vec{u}_1 = \vec{w}_1, \dots, \vec{u}_m = \vec{w}_m$ as desired //

Key special case

$U, W \subseteq V$ subspaces   when is $U + W = U \oplus W$?

Prop $U + W = U \oplus W$ (LHS) $\iff$ $U \cap W = \{\vec{0}\}$ (RHS)

proof LHS $\Rightarrow$ if $\vec{u} \in U \cap W$ $= 0$.

then $\vec{u} = \vec{0} + \vec{u} = \vec{u} + \vec{0}$

$\Rightarrow \vec{0} = \vec{u}, \vec{u} = \vec{0}$. ✓

(trying to prove: $U \cap W = 0$)

RHS $\Rightarrow$ assume $U \cap W = \{\vec{0}\}$

show $U + W$ is a direct sum.

ie: if $\vec{u} + \vec{w} = \vec{0}$, then $\vec{u} = \vec{w} = \vec{0}$.

$\vec{u} = -\vec{w}$ $\therefore \vec{u} \in U \cap W = \{0\}$

$\therefore \vec{u} = \vec{0}$ $\therefore \vec{w} = 0$ //

Polynomials

Let $P(\mathbb{F})$ = set of all polynomials
in a variable, say x,
with coefficients in $\mathbb{F}$.

e.g: $f = a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d$

$g = b_0 + b_1 x + b_2 x^2 + \dots + b_e x^e$ $\quad e \geq d$

When is $f = g$? $\quad (a_i, b_j \in \mathbb{F})$.

$f = g \iff a_0 = b_0, \dots, a_d = b_d, \; b_{d+1} = \dots = b_e = 0$

e.g: $f = 1 + 2x + 5x^3$ (don't write out terms with 0 coeff.)

$= 1 + 2x + 0x^2 + 5x^3$

$= 1 + 2x + 5x^3 + 0 \cdot x^4 + 0 \cdot x^5$

add polynomials: $f + g$

$cf \qquad c \in \mathbb{F}$

$\vec{0}_{P(\mathbb{F})} = 0$ polynomial

<u>prop</u> $P(\mathbb{F})$ is a vector space over $\mathbb{F}$.

subspace

$$P_d(\mathbb{F}) = \{ f \in P(\mathbb{F}) : \text{degree}(f) \leq d \}$$

$$\text{degree}(a_0 + a_1 x + \cdots + a_d x^d) = d.$$

if $a_d \neq 0$ * ← zero poly $a_0 = 0$

$\text{degree}(3) = 0$
← as poly.

$\text{degree}(0) = \underbrace{-\infty}_{}$ $= \max\{i : a_i \neq 0\}$
use this convention.

$P_d(\mathbb{F})$ is a subspace of $P(\mathbb{F})$.

have: $\mathbb{F}^n$

$P(\mathbb{F})$

$\mathbb{F}^{m \times n}$

$\text{Fun}(X, \mathbb{F})$

$\mathbb{F}^\infty$

almost all vector spaces are subspaces of these.

Span, linear independence
bases, dimension.

for all this: Let V be a vector space over $\mathbb{F}$.

Def span: let $S \subseteq V$ be a subset
let $(\vec{v}_1, \dots, \vec{v}_m)$ be a list of vectors in V
(finite!)
ordered!

(a) $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m) = \{ a_1\vec{v}_1 + \dots + a_m\vec{v}_m : a_1, \dots, a_m \in \mathbb{F} \}$
$\subseteq V$

(b) $\operatorname{span} S = \{ a_1\vec{v}_1 + \dots + a_m\vec{v}_m : \begin{matrix} m \geq 1 \\ a_1, \dots, a_m \in \mathbb{F} \\ \vec{v}_1, \dots, \vec{v}_m \in S \end{matrix} \}$

(c) S (or $(\vec{v}_1, \dots, \vec{v}_m)$) spans V if (surprise!), $\operatorname{span} S = V$.
$\left(\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m) = V \right)$.

$\operatorname{span}() = 0 := \{\vec{0}\}$

$\operatorname{span} \phi = 0$.

meta-note:

given a new concept: ask yourself:

(a) why are we defining this?

(b) what are some examples, non-examples?

(c) what is true about this notion?
ie: what theorems / lemmas hold?

play with the concept
make it your friend!

examples

(a) find elements which span $(\mathbb{Z}_5)^3 = \mathbb{F}^3$

(b) same question for $P(\mathbb{F})$, $P_3(\mathbb{F})$

breakout rooms

(a): $\begin{pmatrix}1\\0\\4\end{pmatrix} \in (\mathbb{Z}_5)^3$

$$\operatorname{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = (\mathbb{Z}_5)^3.$$

(b) $P(\mathbb{F}) = \operatorname{span}(\{1, x, x^2, x^3, \ldots\})$

$P_3(\mathbb{F}) = \operatorname{span}(1, x, x^2, x^3)$

theorem $S \subseteq V$

1. span $S \subseteq V$ is a subspace
2. span S is the smallest subspace containing S.

prove this yourself.