Lecture # 7

Math 4310
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Lecture # 7

Last time: $\text{span}(\vec{v}_1, \dots, \vec{v}_m) \subseteq V$

Linear independence

Def The list $(\vec{v}_1, \dots, \vec{v}_m)$ of vectors in V

(a) is called linearly dependent (LD) if

$\exists\, a_1, \dots, a_m \in \mathbb{F}$ NOT ALL ZERO

s.t. $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$ (*)

(b) is called linearly independent (LI) if

whenever $a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = \vec{0}$

then $a_1 = a_2 = \cdots = a_m = 0$.

Similarly, a set S is LI or LD

(define this yourself!)

remarks

- ( ) empty list

( ) is declared to be LI.

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examples

(a) $\left(\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \right)$ in $(\mathbb{F}_5)^3$

this is LD since $3\cdot\begin{pmatrix}1\\0\\3\end{pmatrix} + \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

[also $\begin{pmatrix}2\\0\\1\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\\3\end{pmatrix}$].

(b) in $P_d(\mathbb{F})$ $\quad (1, x, x^2, \ldots, x^d)$

is this LD or LI?

but: $1 - 2x + x^2 = 0$ has solution $x = 1$.

we only consider:

would need for LD:

$1 - 2x + x^2 =$ 0-polynomial

(c) $(0, 1, x, x^2)$ LD or LI?

$$\underbrace{1}_{a_1 \neq 0_{\mathbb{F}}}\cdot 0_V + \underbrace{0_{\mathbb{F}}}_{a_2}\cdot 1 + \underbrace{0}_{a_3}\cdot x + \underbrace{0}_{a_4}\cdot x^2 = \underline{0_V}$$

$\therefore$ LD

$V = P_d(\mathbb{F})$

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(d) consider in breakout rooms:

- $(\sin x, \sin 2x) \in \mathrm{Fun}(\mathbb{R}, \mathbb{R})$

is this LD or LI?

LI

$$\boxed{a \sin x + b \sin 2x = 0} \quad \forall x$$

$$a \sin x + 2b \sin x \cos x = 0$$

$$(\sin x)(a + 2b \cos x) = 0 \qquad \forall x$$

a, b not both 0:

$x = \pi/2$ $\qquad a \cdot 1 + b \cdot 0 = 0 \Rightarrow a = 0$

$x = \frac{\pi}{4}$ $\qquad b \cdot \sin\left(\frac{\pi}{2}\right) = b = 0 \quad \therefore$ LI

- $(\sin x, \sin 2x, \cos x \sin x)$

same question.

since $\sin 2x = 2 \cos x \sin x$

$$0 \cdot \sin x + 1 \cdot \sin 2x + (-2) \cos x \sin x$$

$= 0$ function

$\therefore$ LD

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Some more defs

(a) if $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$

then we call V finite dimensional

(or finitely generated)

(b) if $(\vec{v}_1, \dots, \vec{v}_m)$ spans V, and is LI

then we call $(\vec{v}_1, \dots, \vec{v}_m)$ a basis of V.

(c) if $S \subseteq V$, S is called a basis of V

if (i) $\text{span}\, S = V$

(ii) S is LI.

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Lemma (LI lemma)

Suppose $(\vec{v}_1, \dots, \vec{v}_m)$ is LD in V

then $\exists\ 1 \le j \le m$ s.t.

ⓐ $\vec{v}_j \in \operatorname{span}(\vec{v}_1, \dots, \vec{v}_{j-1})$

and ⓑ removing $\vec{v}_j$ from $(\vec{v}_1, \dots, \vec{v}_m)$

i.e.: $(\vec{v}_1, \dots, \hat{\vec{v}}_j, \dots, \vec{v}_m)$

remove

results in a list with the same span

$\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m)$

if $m=0$ ✓
$()$ is LI

if $m=1$ ✓
ⓐ $\vec{v}_1 \in \operatorname{span}()$
$= 0$

proof $\exists\ a_1, \dots, a_m \in \mathbb{F}$ s.t.

$a_1\vec{v}_1 + \dots + a_m\vec{v}_m = \vec{0}$, not all a_i's are 0.

Let j = largest elem of $1..m$ s.t. $a_j \neq 0$.

then $a_j \neq 0$, $a_{j+1} = \dots = a_m = 0$

$$\vec{v}_j = -\frac{a_1}{a_j}\vec{v}_1 - \frac{a_2}{a_j}\vec{v}_2 - \dots - \frac{a_{j-1}}{a_j}\vec{v}_{j-1}$$

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to paraphrase:

$\vec{v}_j$ is "stoopid".

proves ⓐ.

ⓑ ?? show $\text{span}(\vec{v}_1, \ldots, \hat{\vec{v}_j}, \ldots, \vec{v}_m)$ "LHS"

$= \text{span}(\vec{v}_1, \ldots, \vec{v}_m)$. = RHS

LHS $\subseteq$ RHS "clear"

RHS $\subseteq$ LHS

let $\vec{u} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \in$ RHS

show $\vec{u} \in$ LHS

$$\vec{u} = c_1\vec{v}_1 + \cdots + c_j\left(\frac{-a_1}{a_j}\vec{v}_1 - \cdots - \frac{a_{j-1}}{a_j}\vec{v}_{j-1}\right) + c_{j+1}\vec{v}_{j+1} + \cdots + c_m\vec{v}_m$$

$\in \text{span}(\vec{v}_1, \ldots, \hat{\vec{v}_j}, \ldots, \vec{v}_m)$ //

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Does this handle all cases?!

what about $m = 0$?!

Key theorem Suppose V is finite dimensional

and (a) $(\vec{u}_1, \ldots, \vec{u}_m)$ LI

(b) $(\vec{w}_1, \ldots, \vec{w}_n)$ span V

then $m \leq n$.

try proving this.