proof of key theorem

how to do it?

one way (step 0.): $\mathcal{B} = (\vec{w}_1, \ldots, \vec{w}_n)$ (spans V)

at step 1: remove one of these, put in $\vec{u}_1$:

$\mathcal{B} = (\vec{u}_1, \text{n-1 of the } \vec{w}\text{'s})$ size $\leq n$

2: put in $\vec{u}_2$, remove one of w's

$\mathcal{B} = (\vec{u}_1, \vec{u}_2, \text{(n-2) of the w's})$ size $\leq n$

⋮

m: $\mathcal{B}$ put in $\vec{u}_m$, take out a w. size $\leq n$

$\therefore$ $m \leq n$. since $\mathcal{B}$ includes all u's!

proof:

step 0: $\mathcal{B} = (\vec{w}_1, \ldots, \vec{w}_n)$. spans V, length n.

step 1: $(\vec{u}_1, \underbrace{\vec{w}_1, \ldots, \vec{w}_n}_{\text{span} V})$ whole thing: LD.

remove one from this list using LI lemma.

note: it will be one of $\vec{w}$'s which is removed. $\mathcal{B} = (\vec{u}_1, \text{one of these w's removed})$

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(2)

$\mathcal{B} = (\vec{u}_1, \underbrace{\vec{w}_1, \ldots, \vec{w}_{n-1}}_{\text{renamed.}})$

n elements, spans V.

[if no w's left here: c!] renamed

$\vdots$

step j : start with $\mathcal{B} = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{j-1}, \underbrace{\vec{w}_j, \ldots, \vec{w}_n}_{\text{(from original)}})$

(this works as long as $j \le m$)

needs a $\vec{w}$ to remove!

$\therefore (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{j-1}, \vec{u}_j, \vec{w}_j, \ldots, \vec{w}_n)$ LD!

$\therefore$ LI lemma removes first one in span of previous ones will remove one of $\vec{w}_j, \ldots, \vec{w}_n$.

$\mathcal{B} := (\vec{u}_1, \ldots, \vec{u}_j, \underbrace{\vec{w}_{j+1}, \ldots, \vec{w}_n}_{\text{renamed, but part of original.}})$

after m steps, what do we have?

$\mathcal{B} = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \text{some } w\text{'s})$ length n $\therefore m \le n$.

if no w's left then $\mathcal{B} = (\vec{u}_1, \ldots, \vec{u}_m)$ length n

$\therefore m = n$.

//

$\boxed{\text{theorem}}$ If $(\vec{v}_1, \dots, \vec{v}_n)$ spans V

then: some sublist is a basis of V.

$\boxed{\text{proof}}$ start with $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ assume $\mathcal{B}$ spans V.

<u>step 1</u>: if $\vec{v}_1 = \vec{0}$, then delete $\vec{v}_1$ from $\mathcal{B}$

(otherwise no change to $\mathcal{B}$)

<u>step 2</u>: if $\vec{v}_2 \in \text{span}(\vec{v}_1)$, delete $\vec{v}_2$ from $\mathcal{B}$

step j: if $\vec{v}_j \in \text{span}(\vec{v}_1, \dots, \vec{v}_{j-1})$ delete from $\mathcal{B}$

(otherwise no change)

after n steps, we have a list that still spans V, but LI-lemma $\Rightarrow$ each $\vec{v}_j \notin$ span(ones before it)

(from each step)

$\therefore \mathcal{B}$ LI //

<u>corollary</u> Every finite dimensional vector space has a basis

<u>theorem</u> <u>Any</u> vector space has a basis S.

(uses Zorn's lemma and the axiom of choice)