Question while waiting:
Find a basis for $\mathbb{F}^{2\times 2}$

answer: $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$\dim \mathbb{F}^{2\times 2} = 4$

Last time:

algorithm

if $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{w}_1, \ldots, \vec{w}_m)$

suppose ① $(\vec{v}_1, \ldots, \vec{v}_n)$ is LI.

② $\mathcal{B}$ spans V.

then $\mathcal{B}$ can be pruned down to a basis: $(\vec{v}_1, \ldots, \vec{v}_n$, some (≥ 0) of the $\vec{w}$'s$)$

$\boxed{\text{corollary}}$ If V is finite dimensional then every LI list of vectors (or set) can be extended to a basis of V

proof suppose $(\vec{v}_1, \dots, \vec{v}_n)$ is LI

Let $(\vec{w}_1, \dots, \vec{w}_m)$ span V (exists since V is finite dimensional).

$\therefore\ \exists\ \mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n$, some (≥ 0) of the $\vec{w}$'s $)$ is a basis of V //

$\boxed{\text{theorem}}$ Suppose $(\vec{v}_1, \dots, \vec{v}_m)$ is a basis of V

Suppose $(\vec{w}_1, \dots, \vec{w}_n)$ is another basis of V

then $m = n$

$\boxed{\text{proof}}$ since # LI set $\leq$ # spanning list

v's w's

$\therefore\ m \leq n$

v's span, w's LI

$\therefore\ n \leq m$

$\therefore\ n = m$ //

Def Suppose V is finite dimensional. Define $\underline{\dim V = \dim_{\mathbb{F}} V}$ to be the # elements in any basis.

If V is not finite dimensional, let $\underline{\dim V = \infty}$.

theorem If $\dim V = n$ then

(a) if $(\vec{v}_1, \dots, \vec{v}_n)$ is LI, it is a basis.

(b) if $(\vec{v}_1, \dots, \vec{v}_n)$ spans V, it is a basis.

Examples find basis + dim.

(1) $V = \mathbb{F}^n$

$\mathbb{F}^3$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\mathbb{F}^n: \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$

this is called the <u>standard</u> <u>basis</u> of $\mathbb{F}^n$.

$$\boxed{\dim \mathbb{F}^n = n}$$

2. $P(\mathbb{F})$

$$\{1, x, x^2, x^3, \dots, x^n, \dots\}$$

$$\boxed{\dim P(\mathbb{F}) = \infty}$$

d integer ≥ 0.

3. $P_d(\mathbb{F})$ (polynomials of degree $\leq d$)

basis: $(1, x, x^2, \dots, x^{d-1}, x^d)$

note: this is LI

this spans.

$$\therefore \boxed{\dim P_d(\mathbb{F}) = d+1}$$

$\dim P_0(\mathbb{F}) = 1$

[basis is the poly. "1". $f(x)=1$.]

4. $\dim \{\vec{0}_V\} = 0$.

5 $\dim_{\mathbb{R}} \mathbb{C} = 2$.

($\mathbb{C}$ considered as a vector space over $\mathbb{R}$).

basis : $(1, i)$

$\dim_{\mathbb{C}} \mathbb{C} = 1$ basis : (1)

$\dim_{\mathbb{F}} \mathbb{F} = 1$ basis : (1)

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think about: 1 basis of $\mathbb{F}^{m \times n}$, dim?

2 if X = finite set of size n

what is

$\dim \mathrm{Fun}(X, \mathbb{F})$?

$\boxed{\text{Application}}$ Lagrange interpolation

(FIS section 1.6) "Joe's interpolation method".

Joseph-Louis Lagrange 1795

Waring 1779 did this first

Euler 1783 $\rightsquigarrow$ implies this easily.

Situation
- Let $\mathbb{F}$ be a field with at least $n+1$ elements (fixed $n \geq 1$)
- Let $c_0, c_1, \ldots, c_n \in \mathbb{F}$ be distinct
- Let $d_0, d_1, \ldots, d_n \in \mathbb{F}$ any elements

Question Can we find a polynomial $g \in P_n(\mathbb{F})$ s.t. $g(c_0) = d_0$, $g(c_1) = d_1$, $\ldots$, $g(c_n) = d_n$? If so, is it unique?

Example $P_2(\mathbb{Q})$

$g(0) = 1$
$g(1) = 4$
$g(2) = 11$

$g(x) = 1 + x + 2x^2$

basis of $P_n(\mathbb{F})$: $(1, x, x^2, \ldots, x^n)$

dim $= n+1$

some interesting elements in $P_n(\mathbb{F})$: $0 \le i \le n$

$$f_i(x) = \frac{(x-c_0)(x-c_1)\ldots(x-c_{i-1})(x-c_{i+1})\ldots(x-c_n)}{(c_i-c_0)(c_i-c_1)\ldots(c_i-c_{i-1})(c_i-c_{i+1})\ldots(c_i-c_n)}$$

$$= \prod_{\substack{j=0..n \\ j \neq i}} \frac{x-c_j}{c_i-c_j} \qquad \deg f_i(x) = n.$$

$$f_i(c_j) = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}$$

think about the following

1. is $(f_0, \ldots, f_n)$ LI?
2. does $(f_0, \ldots, f_n)$ span $P_n(\mathbb{F})$

we assumed the $c_0, \ldots, c_n$ are distinct.

why?