Lecture #20 Math 4310

Polynomials

19 oct 2020
Lecture #20 ①

Remarks

① Office hours today: by appointment only

② HW: by end of today, next HW will be posted.
2 HW's combined: due next week

Polynomials

① Division algorithm

for integers: if p, s are non-negative integers
$s \neq 0$

then $\exists!$ non-negative integers r, q s.t.

ⓐ $p = q \cdot s + r$ and

ⓑ $0 \le r < s$

("q": quotient, "r": remainder).

We want the same for polynomials

example suppose $p = x^3+1$, $s = x^2+1$

divide p by s, find q, r.

soln

$$
\begin{array}{r l}
 & \; x \\ \cline{2-2}
x^2 + \quad 1 & \big) \; x^3 + \qquad\qquad\qquad 1 \\
 & -(x^3 + \qquad x) \\ \cline{2-2}
 & \qquad\qquad -x \quad + \quad 1
\end{array}
$$

have:

$$\underset{\overset{\shortparallel}{p}}{x^3+1} = \underset{\overset{\shortparallel}{q}}{x} \cdot \underset{\overset{\shortparallel}{s}}{(x^2+1)} + \underset{\overset{\shortparallel}{r}}{(-x+1)}$$

theorem Division algorithm

Suppose $p, s \in \mathcal{P}(\mathbb{F})$, $s \neq 0$.

then $\exists!$ polynomials $q, r \in \mathcal{P}(\mathbb{F})$ s.t.

ⓐ $p(x) = q(x)\, s(x) + r(x)$

ⓑ $\deg r < \deg s$.

cool proof from Axler Let $n = \deg p$
$m = \deg s$.

if $m > n$, then $q(x) = 0$, $r(x) = p(x)$ is the only solution

$\therefore$ assume $m \le n$.

define

$$T : \overbrace{\underline{P_{n-m}}(\mathbb{F}) \times P_{m-1}(\mathbb{F})}^{\text{dim : } (n-m+1)+m = n+1} \longrightarrow P_n(\mathbb{F}) \quad \text{dim} = n+1$$

$\ni p(x)$

$$T\big(q(x),\ r(x)\big) = s(x)\,q(x) + r(x)$$

in breakout rooms :

ⓐ T is linear ? yes it is.

ⓑ ker T ? im T ?

degree m ↓

$$\ker T = \Big\{ \big(q(x), r(x)\big) : \underbrace{s(x)\,q(x)}_{\substack{\text{degree} \\ \geq m}} + \overset{\uparrow}{r(x)} = 0 \Big\}$$

$\deg \leq m-1$

$\Rightarrow q(x) = 0$

$\therefore r(x) = 0$

$= 0.$

$\therefore$ T is an isomorphism

$\therefore$ surjective.

$$\therefore \exists!\, \big(q(x), r(x)\big) \in P_{n-m}(\mathbb{F}) \times P_{m-1}(\mathbb{F}) \text{ s.t.}$$

$$p(x) = s(x)\,q(x) + r(x).$$

↑

T is inj + surj!

Roots (zeros) of polynomials

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proposition Suppose $p(x) \in \mathcal{P}(\mathbb{F})$, and $\lambda \in \mathbb{F}$.

then $p(\lambda) = 0$ (in $\mathbb{F}$) $\iff$ $p(x) = (x-\lambda)q(x)$ for some $q(x)$.

proof $\Longleftarrow$ this direction is immediate.

$\Longrightarrow$ divide $p(x)$ by $x-\lambda$. $= s(x)$ $\underline{\deg s = 1}$

get $\exists!\ q(x), r(x)$ s.t.

- $p(x) = (x-\lambda)q(x) + r(x)$
- $\deg r(x) \leq 0$

ie: $r(x) = \text{constant}$, or 0

$= r$

plug in $x=\lambda$:

$$\begin{array}{ccccc} p(\lambda) & = & (\lambda-\lambda)q(x) & + & r \\ \parallel & & & & \parallel \\ 0 & & & & 0+r \\ \text{(hypothesis)} & & & & \end{array}$$

$\therefore\ r = 0$

$\therefore\ p(x) = (x-\lambda)q(x)$ //

Corollary For any field $\mathbb{F}$,

if $m = \deg p(x) \geq 0$ (ie: $p(x)$ not identically 0)

then
$$\underset{\#\{\lambda \in \mathbb{F} \,:\, p(\lambda)=0\}}{\underset{\shortparallel}{\#\text{ zeros of } p(x)}} \leq \deg p(x) = m.$$

proof induction on $m = \deg p(x)$.

if $m=0$, then $p(x) = \underset{\mathbb{F}}{\underset{\in}{a}} \neq 0$ has no roots: $\#\text{zeros} \leq 0$ ✓

suppose statement is true for

$\deg = m-1$, show true for m.

consider $p(x)$ of degree m.

suppose λ is a zero.

$\therefore\ p(x) = (x-\lambda)q(x)$ $\deg q(x) = m-1$

$\therefore$ zeros of $p = \{\lambda\} \cup \{\text{zeros of } q(x)\}$

$\therefore\ \#$ zeros of $p \leq 1 + m - 1 = m$ //

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Fundamental theorem of Algebra

Every non-zero polynomial in $P(\mathbb{C})$ has a zero.

<u>proof</u> in an analysis class. /

we will assume this, but not prove it.

if $p(x)$ has degree 2 : quadratic formula
3 : $\exists$ formula
4 : $\exists$ bigger mess of a formula.

Galois : proved, while he was a teenager, that there is <u>no</u> formula in degree 5 or higher

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Corollary if $p(x) \in P(\mathbb{C})$ of degree m, then $\exists!\ \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$ (up to ordering) $a \in \mathbb{C}$

s.t.

$$p(x) = a(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m).$$

(unique up to order of the factors.

proof (See Axler pg. 125, FIS appendix D. /