Lecture #29 **Math 4310**

9 Nov 2020
Lecture #29

Gram-Schmidt
orthogonal complements

My office hours this week:
Monday 3:45 – 4:45 pm
Wed 9 – 10 pm

Situation: V = inner product space ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)
(finite dim'l).

Question: can we find a basis of orthonormal vectors?

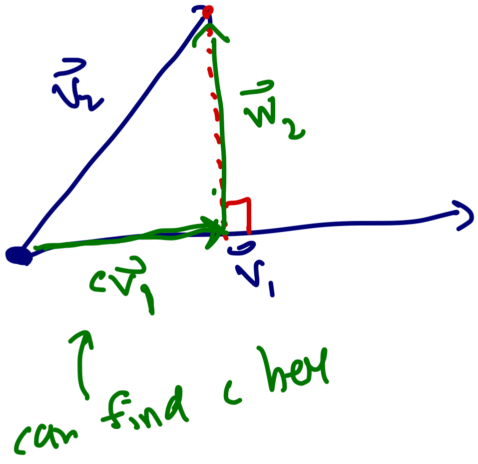
example Consider $2 \overset{LI}{\text{vectors}}$ $\vec{v}_1, \vec{v}_2$ in some V

$W = \mathrm{span}(\vec{v}_1, \vec{v}_2)$. $(\vec{w}_1, \vec{w}_2)$

ⓐ find a basis of W of orthog. vectors

ⓑ find an orthonormal basis of W.

soln picture



$\vec{w}_1 = \vec{v}_1$

find $\vec{w}_2$ in span W, $\perp$ to $\vec{w}_1 = \vec{v}_1$.

find $\vec{w}_2$: $\vec{w}_2 = \vec{v}_2 - c\vec{v}_1$ holds

and $\underline{\langle \vec{w}_2, \vec{v}_1 \rangle} = 0$ $\underline{\text{find } c!}$

$$0 = \langle \vec{w}_2, \vec{v}_1 \rangle = \langle \vec{v}_2 - c\vec{v}_1, \vec{v}_1 \rangle$$

$$= \langle \vec{v}_2, \vec{v}_1 \rangle - c\langle \vec{v}_1, \vec{v}_1 \rangle$$

solve for c: $c = \dfrac{\langle \vec{v}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}$. (note $\langle \vec{v}_1, \vec{v}_1 \rangle > 0$).

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{w}_1 \qquad (\vec{w}_1 = \vec{v}_1)$$

then $(\vec{w}_1, \vec{w}_2)$ is an orthog basis of W.

ⓑ make this orthonormal $\qquad \|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$

$$\frac{\vec{w}_1}{\|\vec{w}_1\|}, \quad \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

Jazz this up:

theorem (Gram-Schmidt)

Let V be an inner product space

$S := (\vec{v}_1, \dots, \vec{v}_n)$ is a LI set in V.

Define a new set $S' = (\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n)$ as follows: ① $\vec{w}_1 = \vec{v}_1$

② $$\vec{w}_k = \vec{v}_k - \left[\frac{\langle \vec{v}_k, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 + \dots + \frac{\langle \vec{v}_k, \vec{w}_{k-1} \rangle}{\langle \vec{w}_{k-1}, \vec{w}_{k-1} \rangle} \vec{w}_{k-1} \right]$$

for $k = 2, 3, \dots, n$.

then S' is an orthogonal set of non-zero vectors s.t.

ⓐ $\operatorname{span} S = \operatorname{span} S'$

ⓑ $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_k) = \operatorname{span}(\vec{w}_1, \dots, \vec{w}_k)$ for all $1 \leq k \leq n$.

idea of proof:

construct $\vec{w}_1 = \vec{v}_1$ ✓

$\vec{w}_2 \perp \vec{w}_1$ (in span(v_1, v_2))

$\vec{w}_3 \perp (\vec{w}_1, \vec{w}_2)$ (in span(v_1, v_2, v_3))

$\vdots$

see also $\vec{w}_k \neq 0$, ⓑ above holds.

proof induction, on #S.

Show:
① span S_k = span S_k'
② S_k is an orthog set of (non-zero) vectors

Let $S_k := (\vec{v}_1, \dots, \vec{v}_k)$

$S_k' := (\vec{w}_1, \dots, \vec{w}_k)$

if $k = 1$

$S_1 = S_1' = (\vec{v}_1)$ $\vec{v}_1 \neq 0$. (since S_k LI).

assume stmt in red is true for $k-1$

span S_{k-1} = span S_{k-1}'

S_{k-1}' is a bunch of orthog non-zero vecs.

show same is true for S_k. ✪

$S_k' = \{\vec{w}_1, \dots, \vec{w}_k\}$

know $\langle \vec{w}_i, \vec{w}_j \rangle = 0$ $1 \leq i, j \leq k-1$

also know $\vec{w}_1, \dots, \vec{w}_{k-1} \neq 0$.

NTS: $$\begin{cases} \langle \vec{w}_k , \vec{w}_i \rangle = 0 \quad \forall\ \underline{1 \le i \le k-1}. \\ \vec{w}_k \neq \vec{0}. \end{cases}$$

by def

$$\vec{w}_k = \left\{ \underline{\vec{v}_k} - \left[\frac{\langle \vec{v}_k , \vec{w}_1 \rangle}{\langle \vec{w}_1 , \vec{w}_1 \rangle} \vec{w}_1 + \cdots + \frac{\langle \vec{v}_k , \vec{w}_i \rangle}{\langle \vec{w}_i , \vec{w}_i \rangle} \vec{w}_i + \cdots + \frac{\langle \vec{v}_k , \vec{w}_{k-1} \rangle}{\langle \vec{w}_{k-1} , \vec{w}_{k-1} \rangle} \vec{w}_{k-1} \right] \right\}$$

$$\langle \vec{w}_k , \vec{w}_i \rangle = \underline{\langle \vec{v}_k , \vec{w}_i \rangle} - \frac{\langle \vec{v}_k , \vec{w}_i \rangle}{\langle \vec{w}_i , \vec{w}_i \rangle} \underline{\langle \vec{w}_i , \vec{w}_i \rangle}$$

$$= 0 \quad \underline{!!}$$

$\therefore \langle \vec{w}_k , \vec{w}_i \rangle = 0.$

NTS: $\vec{w}_k \neq \vec{0}$

if $\vec{w}_k = \vec{0}$, then $\vec{v}_k \in \operatorname{span}(\vec{w}_1, \ldots, \vec{w}_{k-1})$
$$\| \quad \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{k-1})$$

but $v_k \notin \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{k-1})$ since $\{\vec{v}_1, \ldots, \vec{v}_k\}$ L.I.

∴ $\vec{w}_k \neq \vec{0}$

∴ shown $\vec{w}_k \neq 0$ $\perp$ to all before.

$$\operatorname{span}(w_1,\ldots,w_k) = \operatorname{span}(v_1,\ldots,v_k)$$

$$\| \qquad\qquad\qquad // \leftarrow \text{think about this prove for yourself.}$$

$$\operatorname{span}(\vec{v}_1,\ldots,\vec{v}_{k-1},\vec{w}_k)$$

Example Find orthog. basis (and orthonormal basis

for $V = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \subseteq \mathbb{R}^4$

$\qquad\qquad\qquad \vec{v}_1 \qquad \vec{v}_2 \qquad \vec{v}_3$

Soln $\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\langle \vec{w}_1, \vec{w}_1 \rangle = 2$

$v_2 \cdot w_1 = v_2 \cdot v_1 = 1.$

$$\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} = \vec{v}_2 - \left(\frac{v_2 \cdot w_1}{w_1 \cdot w_1} \right) \vec{w}_1 = \vec{v}_2 - \frac{1}{2} \vec{w}_1 .$$

$\vec{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$ (can take $\vec{w}_2 =$

or any non-zero multiple we want)

$$\vec{w}_3 = \vec{v}_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1}\vec{w}_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2}\vec{w}_2$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}$$

or $\vec{w}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}$ $\quad (3 \cdot \quad).$

orthonormal basis:

$$\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix} \right).$$

Def $W \subseteq V$ is a subspace

$$W^{\perp} := \{ \vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \quad \forall \vec{w} \in W \}.$$

orthogonal complement of W

$$\left.\begin{array}{l}\bullet\quad W^{\perp} \subseteq V \quad \text{is a subspace} \\ \bullet\quad W \cap W^{\perp} = 0\end{array}\right\}$$