Practice problems # 1
Math 4310 Fall 2020

Problems.

1. True/False. Determine if the statement is logically true or logically false. Clearly write the full word True, or False, and box it. Give a short (no more than 5 sentence) justification for your answer.

   (a) Let U, W_1, W_2 be subspaces of a vector space V. Then necessarily,
   $$U \cap (W_1 + W_2) = U \cap W_1 + U \cap W_2.$$

   (b) The subset
   $$U_1 = \{T \in \mathcal{L}(V) \mid T^2 = T\}$$
   is a subspace of $\mathcal{L}(V)$.

   (c) The subset
   $$U_2 = \{f \in \mathcal{F}un(\mathbb{R}, \mathbb{R}) \mid f(x) = f(1-x), \forall x \in \mathbb{R}\}$$
   is a subspace of $\mathcal{F}un(\mathbb{R}, \mathbb{R})$.

   (d) The map $F : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ defined by
   $$F(f(x)) = \sin(x)f(x)$$
   is a linear map.

   (e) The map $T : \mathcal{P}(\mathbb{F}) \longrightarrow \mathcal{P}(\mathbb{F})$ defined by
   $$T(p(x)) = x^2 p(x^2)$$
   is a linear map.

   (f) The subset $I \subset \mathbb{R}$ consisting of the irrational numbers is a field.

2. Short answer. Give a short (no more than 5 sentence) justification for your answer.

   (a) Let $T \in \mathcal{L}(V)$ be a linear map, where V is finite dimensional. Suppose that $\mathrm{im}(T) \neq \mathrm{im}(T^2)$. Then which of the following is true?

      i. T must be invertible.
      ii. T must be singular (i.e. not invertible).
      iii. T could be either invertible or not invertible, it depends on the specific T.

3. Find the dimensions of the following vector spaces.

(a) $V_1 = \mathcal{F}un(\{a, b, c, d\}, \mathbb{F})$.

(b) $V_2 = \mathcal{L}(\mathbb{C}^2, \mathbb{C}^{13})$ (over $\mathbb{F} = \mathbb{C}$).

(c) $V_3 = \mathcal{F}un(\mathbb{F}^3, \mathbb{F})$, where $\mathbb{F} = \mathbb{Z}_2$ is the field with 2 elements.

(d) $V_4 = \mathcal{F}un(\mathbb{F}^3, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$.

4. Let $\mathbb{F} = \mathbb{Z}_7$, and suppose that the matrix $A \in \mathbb{F}^{4\times 5}$ has an LU decomposition with

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Let L_A be the linear map given by multiplication by A. Find the dimension of the image of L_A, and find a basis for the kernel of L_A.

(b) Does every system of linear equations $A\mathbf{x} = \mathbf{b}$ have a solution? (A is the matrix we have been considering throughout this problem).

(c) If the system $A\mathbf{x} = \mathbf{b}$ does have a solution, how many solutions does it have? As for all problems, justify your answer!

5. Let V be the vector space $\mathcal{P}(\mathbb{R})$. Let c_1, c_2, c_3 be distinct real numbers. Define the functions $\ell_i : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}$ by $\ell_i(p(x)) = p(c_i)$.

(a) Show that $\ell_i : V \longrightarrow \mathbb{R}$ is linear (for $i = 1, 2, 3$), and therefore an element of the vector space $V^* = \mathcal{L}(V, \mathbb{R})$.

(b) Prove that (ℓ_1, ℓ_2, ℓ_3) is linearly independent (perhaps using Lagrange polynomials would be helpful here).

6. Let $T : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R})$ be a linear map, and suppose we know that

$$\begin{aligned} T(x^2 + 1) &= x^2 - x, \text{ and} \\ T(1) &= 2x + 1. \end{aligned}$$

Given this partial information, answer the following questions (as always, justify your answers).

(a) Could T be injective, given the above data?

(b) Could T be surjective, given the above data?

(c) Can we determine $T(x^2 + x + 1)$, given the above data?

(d) Can we determine whether $x^2 + x + 1$ is in the image of T, given the above data?

(e) Can we determine whether $2x + 1$ is in the kernel of T, given the above data?

7. Suppose that $T : \mathbb{F}^4 \longrightarrow \mathbb{F}^4$ satisfies $T^4 = 0$, but $T^3 \neq 0$. Let $v \in \mathbb{F}^4$ be chosen so that $T^3(v) \neq \vec{0}$

(a) Show that $\mathcal{B} = (v, T(v), T^2(v), T^3(v))$ is a basis for $\mathbb{F}^4$.

(b) Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$

(c) Find the dimensions of the image of T and of the kernel of T.

8. Let $T \in \mathcal{L}(V)$. A subspace $W \subseteq V$ is called T**-invariant** if whever $w \in W$, then $T(w) \in W$, that is, if $T(W) \subseteq W$.

   (a) Show that $\ker(T)$ is T-invariant. Show that $\mathrm{im}(T)$ is T-invariant.

   (b) Suppose that $S, T \in \mathcal{L}(V)$, and that $ST = TS$ (i.e. that S and T commute). Show that $\ker(S)$ is T-invariant. Show that $\mathrm{im}(S)$ is T-invariant.