Math 7670 Combinatorial Structure of Hilbert Schemes

2 Sep 2020
Lecture #1

mechanics:

(a) canvas, website

(b) lunch office hours need to decide time + day

(c) texts place papers on web
texts also.

Fix k = field

Let $S = k[x_0, \ldots, x_n]$ homog coord ring of $\mathbb{P}^n$

If $I \subseteq S$ homog ideal,

let $\mathbb{V}(I) \subseteq \mathbb{P}^n$ be the corresponding subscheme.

Question: which of the following ideals result in same subschemes in $\mathbb{P}^1_{xy}$:

(a) (x) $\qquad \mathbb{V}(x) = \mathbb{V}(x^2, xy)$

(b) (x,y) $\qquad \mathbb{V}(x,y) = \emptyset$

(c) (x^2, xy) $\qquad \mathbb{V}(x^2, xy) = \mathbb{V}(x)$

(d) (x^2) $\qquad \mathbb{V}(x^2) = \mathbb{V}(x^3, x^2y)$

(e) (x^3, x^2y)

$(x^2) \subseteq (x)$ so $\mathbb{V}(x) \underset{\neq}{\subseteq} \mathbb{V}(x^2)$ "thick y-axis"

Def Fix $I \subseteq S$ homog ideal

Let $I^{sat} := \{ f \in S : x_i^N f \in I \ \forall i = 0..n,\ N \gg 0 \}$

$$= (I : (x_0, \dots, x_n)^\infty).$$

$I \subseteq I^{sat}$

- $(x^2, xy)^{sat} = (x)$
- $(x^3, x^2y)^{sat} = (x^2)$

Fact If $I, J \subseteq S$ are homog. ideals, then

$$\underset{\subseteq \mathbb{P}^n}{\mathbb{V}(I)} = \mathbb{V}(J) \iff I^{sat} = J^{sat}.$$

(b) Hilbert functions + polynomials :

Hilbert function of S/I : $\quad h_{S/I}(z) = \dim_k (S/I)_z$

$$h_{S/I} : \mathbb{N} \longrightarrow \mathbb{N}.$$

Hilbert polynomial of S/I : $\exists$ a polynomial $P_{S/I}(z)$

s.t. $\forall z \gg 0$, $P_{S/I}(z) = h_{S/I}(n)$ $\quad$ Hilbert polynomial of I, or $X = \mathbb{V}(I)$.

(notation: $P_X(z) = P_{S/I}(z)$)

facts: ① $\deg P_X = \dim X$

② $P_X(z) = \frac{e}{d!} z^d +$ lower terms

then $d = \dim X$, $e = \deg X$.

in $\mathbb{P}^n$, consider all subschemes
with a fixed Hilbert polynomial. $p(z)$

$$\boxed{\mathrm{Hilb}^{p(z)}(\mathbb{P}^n)} = \{ V(I) : P_{S/I}(z) = p(z) \}$$

as a set!

Amazing thing happens: this has the structure of an algebraic subset of $\mathbb{P}^n$

Even more: it has a natural scheme structure.

prove this has structure

use flatness, functor of points, representability

2 examples

(i) (projective) lines in $\mathbb{P}^3$ $\quad L = V(l_1, l_2)$ $\quad l_1, l_2 \in S$ linear form)

e.g. $V(x_2, x_3)$ $\quad k[x_0,x_1,x_2,x_3]/(x_2,x_3) \cong k[x_0,x_1]$

$P_L(z) = z+1$ Hilbert poly.

$\mathrm{Hilb}^{z+1}(\mathbb{P}^3)$?

answer: "$\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$" $= \mathrm{Gr}(2,4)$

know: $\mathrm{Hilb}^{z+1}(\mathbb{P}^3) = \mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}^5$
quadric hypersurface.

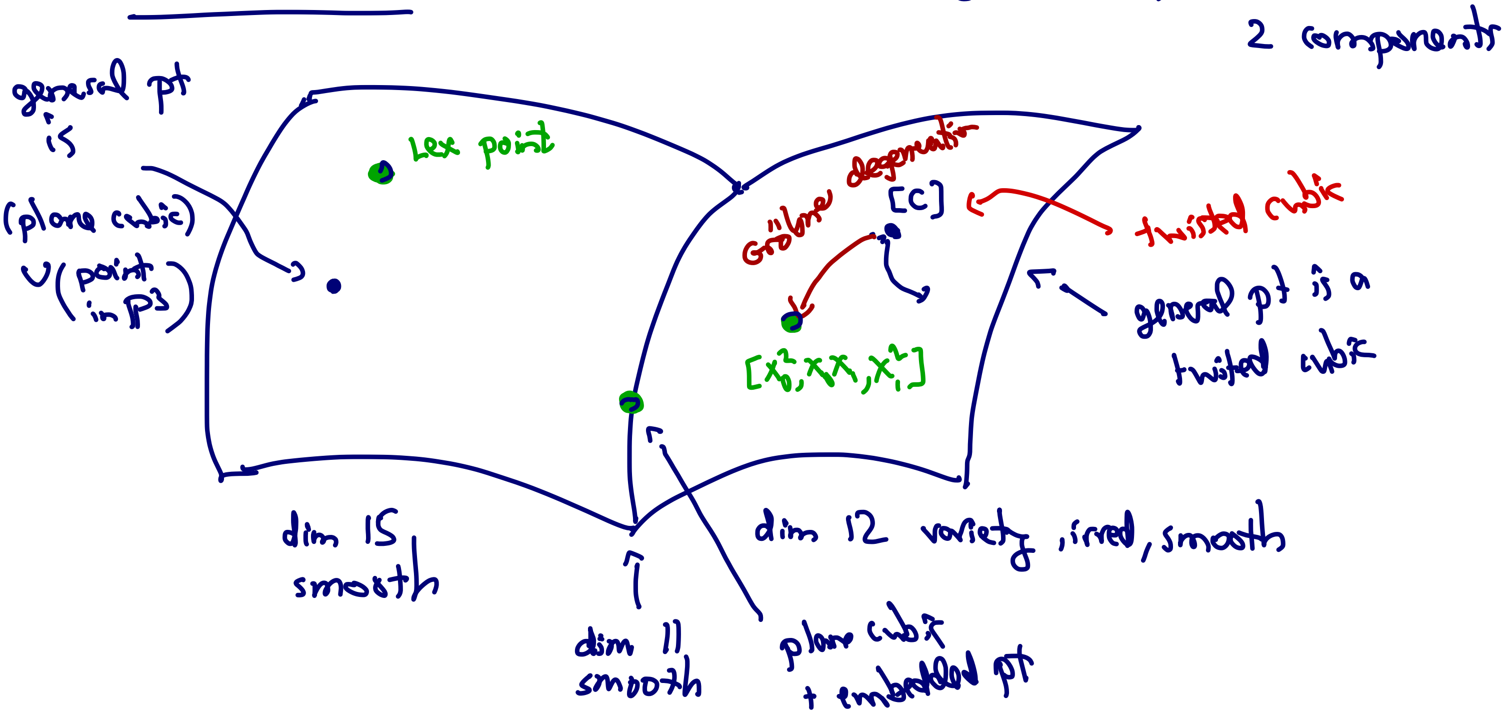
(2) twisted cubic curve in $\mathbb{P}^3$

$$\mathbb{P}^1 \hookrightarrow C \subseteq \mathbb{P}^3$$
$$(s,t) \longmapsto (s^3, s^2t, st^2, t^3)$$

C degree 3, genus 0. $\quad P_C(z) = 3z+1$ ← check!

ideal of C : $I_2\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = I$

<u>$\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$</u> : (Piene–Schlessinger 1983)



- GB ⇝ path on Hilb
- Zariski tangent space of Hilb at $[X]$ $\quad X = V(I)$.
  do it directly from I.
- Borel fixed ideals "subway stops"
- Non-reduced components? What does that mean?
  Mumford example.

2 Sep 2020
Lecture 1

5

- Vakil's theorem : Murphy's Law.
- Lots of open problems
- Hilbert scheme of points
- construction of Grothendieck
  - Regularity
  - Macaulay / Gotzmann
- lexicographic component
  lex. ideal is smooth point
- connectedness (Hartshorne, Reeves)
  "radius".
- equations
  versal deformations
  flatness