Lecture #2

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Before construction of $\mathrm{Hilb}^{P(z)}(\mathbb{P}^n)$ we need some tools:

① Gröbner bases + initial ideals
(ref: Cox, Little, O'Shea)

② Generic initial ideals + Borel-fixed ideals

③ Castelnuovo–Mumford regularity

④ Flat families

⑤ Macaulay / Gotzmann's theorems about Hilb. fcn's, polys.

Gröbner bases example $I = \langle b^2 - ac,\ bc - ad,\ c^2 - bd \rangle$
$\subseteq S = k[a,b,c,d]$

ideal of the twisted cubic curve in $\mathbb{P}^3$

term order on $S = k[x_0, \dots, x_n]$:

this is a total order $>$ on the monomials of S.

[monomials: $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ $\quad \alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$

$\deg(x^\alpha) = |\alpha| = \sum_{i=0}^{n} \alpha_i$.]

s.t. ⓐ $x^\alpha > 1 \quad \forall\ x^\alpha \neq 1$.

ⓑ $x^\alpha > x^\beta \implies x^{\alpha+\gamma} > x^{\beta+\gamma} \quad \forall\ \alpha, \beta, \gamma$.

Examples $I = (b^2-ac,\ bc-ad,\ c^2-bd)$

order	GB	in(I)
ⓐ $a>b>c>d$ grevlex	$\boxed{c^2}-bd,\ \boxed{bc}-ad,\ \boxed{b^2}-ac$	(b^2, bc, c^2)
ⓑ lex order	$\boxed{bd}-c^2,\ \boxed{ad}-bc,\ \boxed{ac}-b^2$	(ac, ad, bd)
ⓒ $b>c>a>d$ lex	$\boxed{b^2}-ac$ $\boxed{bc}-ad$ $\boxed{bd}-c^2$ $\boxed{c^3}-ad^2$	(b^2, bc, bd, c^3)

grevlex on S:

$x^\alpha > x^\beta \iff |\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the last non-zero entry in $\alpha-\beta$ is negative.

By the way: I has 8 different $\mathrm{in}(I)$.

notes ① $\mathrm{in}_>(f) = \mathrm{in}(f) =$ lead term (w. the coeffs.)

② $\mathrm{in}_>(I) = \mathrm{in}(I) = \langle \mathrm{in}(f) : f \in I \rangle$ monomial ideal.

③ $G = \{g_1, \dots, g_r\} \subseteq I$ is a GB of I (wrt $>$) if $\langle \mathrm{in}(g_1), \dots, \mathrm{in}(g_r) \rangle = \mathrm{in}(I)$

property of lex, grevlex

example $x > y > z$

degree 2 monomials

$$x^2 > xy > xz > yz > z^2$$
$$> y^2$$

is = lead term

grevlex: $x^2 > xy > y^2 > \boxed{xz} > yz > z^2$

lex : $x^2 > xy > xz > \boxed{y^2} > yz > z^2$

Prop ⓐ $x_n \mid f \iff x_n \mid \text{in}_{\text{grevlex}}(f)$

ⓑ $f \in k[x_i, \dots, x_n] \iff \text{in}_{\text{lex}}(f) \in k[x_i, \dots, x_n]$.

Hilbert functions, series + polynomials

Situation: M is a (f.g.) graded S-module.

$\mathbb{Z}$-graded, $\left(M = \bigoplus_{z=-\infty}^{\infty} M_z\right)$.

① Hilbert function:

$$h_M(z) := \dim_k M_z . \qquad h_M : \mathbb{Z} \longrightarrow \mathbb{N}$$

twisted cubic

$z=0$

$$h_{S/I}(z) : [1, 4, 7, 10, 13, \dots]$$
$$= \begin{cases} 3z+1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 . \end{cases}$$

② Hilbert series

$$H_M(t) := \sum_{z=-\infty}^{\infty} (\dim_k M_z) t^z .$$

fact: $H_M(t) = \dfrac{Q(t)}{(1-t)^{n+1}}$, $Q(t) \in \mathbb{Z}[t, t^{-1}]$

twisted cubic

$$H_{S/I}(t) = \frac{1-3t^2+2t^3}{(1-t)^4} = \frac{1+2t}{(1-t)^2}$$

③ Hilbert polynomial

$\exists$ poly. P_M s.t. $\forall\, z \gg 0$ $\quad P_M(z) = h_M(z)$

twisted cubic : $P_{S/I}(z) = 3z+1$.

Prop (Macaulay)

If $I \subseteq S$ is homogeneous, then for any term order $>$, $\dim I_z = \dim \mathrm{in}_>(I)_z$.

(in fact - this works for modules)

so : to compute $h_{S/I}$ $\rightsquigarrow$ reduce this to $h_{S/\mathrm{in}(I)}$.

idea of proof of prop?

Let $f_1,\dots,f_r$ be a basis over k of I_z.

order monomials in this degree: $x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_N}$

descending order wrt $>$.

make matrix over k:

$$\begin{array}{c|cccc} & x^{\alpha_1} & x^{\alpha_2} & \cdots & x^{\alpha_N} \\ \hline f_1 & & & & \\ \vdots & & \text{entries in } k & & \\ f_r & & & & \end{array}$$

row-reduce. pivot columns are lead terms, give

$r = \dim I_z = \dim \mathrm{in}(I)_z$.

Computing Hilbert series

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key observation #1 :

given any finite exact sequence of graded S-modules, maps of degree 0 :
(graded)

$$\begin{tikzcd} 0 & C_0 \arrow[l] & C_1 \arrow[l, "\varphi_1"'] & C_2 \arrow[l, "\varphi_2"'] & \cdots \arrow[l] & C_N \arrow[l, "\varphi_N"'] & 0 \arrow[l] \end{tikzcd}$$

then $\sum_{i=0}^{N} (-1)^i h_{C_i}(z) = 0.$

(ie: h_M, H_M, P_M are additive on exact sequences).

important exact seq :

Given $I \subseteq S$ homog, $f \in S_d$, $f \notin I$

then

$$\begin{tikzcd} 0 \arrow[r] & \frac{(I,f)}{I} \arrow[r] \arrow[d, phantom, "\wr\shortparallel"] & S/I \arrow[r] & S/(I,f) \arrow[r] & 0 \\ & S/(I:f)\,(-d) & & & \end{tikzcd}$$

is an exact sequence.

(notation : $M(-d)_z = M_{-d+z}$).

also $I:f = \{ g \in S : gf \in I \} \subseteq S$

example $(a^2, ab^2, b^3c) : ab = (a, b, b^2c)$
$= (a, b).$

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exercise $(x^{\alpha_1}, \dots, x^{\alpha_r}) : x^{\beta} = \left(\frac{x^{\alpha_1}}{\gcd(x^{\alpha_1}, x^{\beta})}, \dots, \frac{x^{\alpha_r}}{\gcd(x^{\alpha_r}, x^{\beta})} \right).$

Algorithm

if $I = \langle x^{\alpha_1}, \dots, x^{\alpha_r} \rangle$ is a monomial ideal:

$$0 \to S/I : x^{\beta}(-\deg x^{\beta}) \longrightarrow S/I \longrightarrow \underbrace{S/I, x^{\beta}}_{\text{hilb fcn}} \longrightarrow 0$$

$$\underbrace{\le r \text{ gens} \qquad\qquad r \text{ gens}}_{\text{can compute these}} \qquad\qquad \underset{\therefore \text{ compute this.}}{r+1 \text{ gens}}$$

exercise use this to find $H_{S/I}(t)$, I = twisted cubic.