$\boxed{\text{Lecture \# 3}}$

$\boxed{\text{7 Sep 2020 Lecture \#3}}$ ①

Generic initial ideals

Borel fixed ideals

Let k = infinite field

$S = k[x_0, \ldots, x_n]$

Let $G = GL_k(n+1)$

G acts on S : $g = (g_{ij}) \in G$

$$\begin{cases} g \cdot x_i = \sum_{j=0}^{n} g_{ij} x_j \\ g \cdot f(x_0, \ldots, x_n) = f(g(x_0), \ldots, g(x_n)) \\ g \cdot I = (g \cdot f : f \in I) \subseteq S. \end{cases}$$

<u>Question</u> Let $I \subseteq S$ be a homog. ideal

Let $>$ be a fixed term order.

as $g \in G$ varies, how does $\mathrm{in}_>(g \cdot I)$ vary?

7 Sep 2020
Lecture #3

theorem (Galligo)

Fix $I \subseteq S$ homog.

let $>$ be a fixed term order

then $\exists$ monomial ideal $J \subseteq S$ and a non-empty Zariski open subset $\mathcal{U} \subseteq G$ s.t. $\forall g \in \mathcal{U}$, $\mathrm{in}_>(g \cdot I) = J$.

affine variety

notation: J is called the generic initial ideal of I, $J =: \mathrm{gin}_>(I)$.

also: if $>$ = grevlex order, $x_0 > x_1 > \cdots > x_n$, say that $J =: \mathrm{gin}(I)$

no order here means grevlex.

Useful construction :-)

Suppose $f_1, \ldots, f_q$ is a basis of I_d s.t. $\mathrm{in}(f_1) > \cdots > \mathrm{in}(f_q)$

$\therefore \mathrm{in}(I)_d = \mathrm{span}(\mathrm{in}(f_1), \ldots, \mathrm{in}(f_q))$

Consider $\Lambda^q S_d$ has as basis monomials

$$\Longrightarrow \quad m = x^{\alpha_1} \wedge x^{\alpha_2} \wedge \dots \wedge x^{\alpha_q} \qquad x^{\alpha_1} > \dots > x^{\alpha_q}$$

$$f := \underbrace{f_1 \wedge \dots \wedge f_q}\ \in \Lambda^q S_d .$$

$$\mathrm{span}_k(f) \overset{1-1}{\longleftrightarrow} \text{subspace } \mathrm{span}_k(f_1, \dots, f_q)$$

<u>notation</u> $w(m) := x^{\alpha_1} x^{\alpha_2} \dots x^{\alpha_q} \in S.$
<u>weight</u>

$$f = f_1 \wedge \dots \wedge f_q$$
$$= \mathrm{in}(f_1) \wedge \dots \wedge \mathrm{in}(f_q) + \text{terms of lower weight.}$$

aside: if $\delta = \begin{pmatrix} \delta_0 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix} \in G$

and $f = \underbrace{\mathrm{in}(f_1) \wedge \dots \wedge \mathrm{in}(f_q)}_{\text{weight } w_0} + \sum_{\substack{w \\ \text{monomials} \\ \text{in } S}} c_w f_w$

then
$$\delta \cdot f = w_0(\delta_0, \dots, \delta_n) + \sum c_w\, w(\delta_0, \dots, \delta_n) f_w$$

where if $w = x_0^{w_0} \cdots x_n^{w_n}$

then $w(\delta) := \delta_0^{w_0} \cdots \delta_n^{w_n}$.

<u>proof of thm</u> fix d for the moment

Let $I_d = \mathrm{span}(f_1, \dots, f_q)$, $\underline{\underline{\mathrm{in}(f_1) > \cdots > \mathrm{in}(f_q)}}$

let $f = f_1 \wedge \cdots \wedge f_q$.

Let $h = (h_{ij})$ be an $(n+1) \times (n+1)$ matrix of indeterminants.

$$h \cdot f = (hf_1) \wedge \cdots \wedge (hf_q)$$

$$= g_{w_0} + g_{w_1} + \cdots$$

each g_{w_i} = sum of monomials of wt w_i.

coeffs are polynomials in h_{ij}'s.

$$h \cdot f = P_d(h_{ij}) \cdot x^{\alpha_1} \wedge \cdots \wedge x^{\alpha_q} + \text{lower wt terms}$$

some poly

det of with cols given by monoms.

7 Sep 2020
Lecture #3

$$\begin{array}{cc} & x^{\alpha_1} \; \cdots \; f \; \cdots \\ \begin{array}{c} h(f_1) \\ \vdots \\ h(f_q) \end{array} & \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array}$$

So if $g \in G$, $P_d(g) \neq 0$

then $\mathrm{in}(g \cdot I)_d = \mathrm{span}(x^{\alpha_1}, \dots, x^{\alpha_q})$

Let $U_d := G \setminus \mathbb{V}(P_d)$

Define $J_d := \mathrm{in}(g \cdot I)_d = \mathrm{span}(x^{\alpha_1}, \dots, x^{\alpha_q})$.

Define $J = \bigoplus_{d=0}^{\infty} J_d \subseteq S$

show 2 things (a) J is an ideal.

(b) $\underbrace{g \in \bigcap_{d \geq 0} U_d}_{\text{want: this is a finite intersection.}} \Rightarrow \mathrm{in}(gI) = J$.

(a) show $S_1 J_d \subseteq J_{d+1}$ $\forall d$.

let $g \in U_d \cap U_{d+1}$, $\mathrm{in}(gI)_d = J_d$
$\mathrm{in}(gI)_{d+1} = J_{d+1}$ ✓

$\therefore$ J is an ideal.

(b) know: J = monomial ideal

let e = max deg. of a generator for J.

for $g \in \mathcal{U}_0 \cap \cdots \cap \mathcal{U}_e$ have:

$$\mathrm{in}(gI)_d = J_d \qquad \forall\, d \le e.$$

$$\therefore \quad J \subseteq \mathrm{in}(g \cdot I)$$

but for any d,

$$\dim I_d = \dim J_d \le \dim (g \cdot I)_d = \dim I_d$$

$$\therefore J = \mathrm{in}(gI) \quad \forall g \in \mathcal{U}_0 \cap \cdots \cap \mathcal{U}_e$$

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proof shows: $\forall g \in G$

if $(x^{\alpha_1}, \ldots, x^{\alpha_q})$ is a basis for $\mathrm{in}(g \cdot I)_d$

then $w(x^{\alpha_1} \wedge \cdots \wedge x^{\alpha_q}) \le w(\mathrm{gin}\, I_d)$

examples generic initial ideals of

twisted cubic

gs: (a^2, ab, b^2)

(a^2, ab, ac, b^3)

(did via Macaulay2: file will be posted).

$\mathrm{gin}_>(I)$ is Borel-fixed

Let $B = \{ g = (g_{ij}) \in G : g_{ij} = 0 \quad i<j \}$

Borel group of lower triangular matrices

includes D = diagonal matrices

let B' = upper triangular matrices.

B is generated by D and for all

$0 \le j < i \le n+1, \quad a \in k$

$$g_{ij}(a) \begin{cases} x_i = x_i + a x_j \\ x_\ell = x_\ell \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 + a x_0 \\ x_2 \end{bmatrix}$$

Suppose $J \subseteq S$ is an ideal, fixed by B

ie: $\forall g \in B$, $g \cdot J = J$.

what can we say about J?

7 Sep 2020
Lecture #3

<u>prop!</u> if $J \subseteq S$ is fixed by D,

then J is a monomial ideal

(recall: k = infinite)

<u>pf</u> exercise.

<u>examples</u> (a) $J = (x^2, y^2)$ is this Borel-fixed?

not Borel fixed: $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \to$ $x \mapsto x$, $y \mapsto x+y$

$g \cdot J = (x^2, (x+y)^2) \neq J$.

Oops: if char $k = 2$, $g \cdot J = (x^2, y^2)$

take home: char $k = p$ is a bit more complicated.

(b) (x^2, xy, y^2) is this B-fixed

(think about that).