last time:

$G = GL(n+1)$

$B \subseteq G$ Borel subgroup of lower triangular matrices

showed: $\exists\ \mathcal{U} \subseteq G$, mon. ideal $J \subseteq S$ s.t. $\forall g \in \mathcal{U}$, $in_>(g \cdot I) = J$ (fixed $>$).

B-fixed ideal

$D \subseteq B$
$\uparrow$ diag. matrices

$\therefore$ B-fixed ideal is a monomial ideal.

examples

a) (x^2, y^2) char $k \neq 2$ not Borel fixed
not strongly stable
char $k = 2$ it is!

b) $(x^2, xy, y^2) = J$ strongly stable

$$g = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \qquad \begin{aligned} g \cdot x &= x \\ g \cdot y &= y + ax \end{aligned}$$

$$g\cdot(xy) = x(y+ax)$$
$$= xy + ax^2 \in J$$
$$g\cdot(y^2) = (y+ax)^2 \in J.$$

J is B-fixed.

Def A monomial ideal $J \subseteq S$ is strongly stable if $\forall\, x^\beta \in J$, if $x_i^a \mid x^\beta$ and $j<i$ then $\left(\frac{x_j}{x_i}\right)^a x^\beta \in J$.

(on exp vectors: $x^{(a_0,\dots,a_n)} \in J \to x^{(a_0,\dots,a_j+m,\dots,a_i-m,\dots a_n)} \in J$) if $a_i \geq m$.

lemma/exercise J is strongly stable if for all generators $x^\beta \in J$, $x_i \mid x^\beta$, then $\frac{x_j}{x_i}\cdot x^\beta \in J$ (if $j<i$).

<u>prop</u> Suppose char $k = 0$

Then a monomial ideal $J \subseteq S$ is B-fixed $\iff$ J is strongly stable.

<u>note</u> for char $k = p > 0$

more difficult, but still ok

see Eisenbud comm alg: 15.9.3.

<u>example</u> $S = k[x_0, \dots, x_3]$ (char $k = 0$)

Is $J = (x_0^2, x_0x_1, x_0x_2, x_1^3)$ B-fixed?

same question for char $k = 2$ or 3?

<u>proof of prop</u>

Leave this as an exercise.

Situation:

$I \subseteq S$ homog ideal

$>$ term order with $x_0 > x_1 > \cdots > x_n$.

k infinite field

__theorem__ (Galligo, Bayer–Stillman)
The generic initial ideal $\mathrm{gin}_>(I)$ is $\mathcal{B}$-fixed.

__idea of the proof__

WLOG assume $\mathrm{in}_>(I) = \mathrm{gin}_>(I) =: J$
(by replacing I by $g \cdot I$, $g \in \mathcal{U}$).

J_d is $\mathcal{B}$-fixed. How to tell this?
$\| \; (x^{\alpha_1}, \ldots, x^{\alpha_q})$
is $\mathcal{B}$-fixed $\iff$ $\mathrm{span}(x^{\alpha_1} \wedge \cdots \wedge x^{\alpha_q})$ is $\mathcal{B}$-fixed.

if $(\star)$ $g = g_{ij}(a) \in \mathcal{B}$, $j < i$ $\quad \begin{cases} x_i \longmapsto x_i + a x_j \\ x_\ell \longmapsto x_\ell \end{cases}$

$$\underline{g(x^{\alpha_1} \wedge \cdots \wedge x^{\alpha_q})} = \underbrace{\text{sum of monomials of weight} > w(x^{\alpha_1} \wedge \cdots \wedge x^{\alpha_q})} + x^{\alpha_1} \wedge \cdots \wedge x^{\alpha_q}$$

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need these terms to be all 0.
in order for J_d to be B-fixed.

Let $I_d = \mathrm{span}(f_1, \dots, f_q)$ (can suppose $\underline{in(f_i) = x^{\alpha_i}}$).

Let $f = f_1 \wedge \dots \wedge f_q$

$$= x^{\alpha_1} \wedge \dots \wedge x^{\alpha_q} + \sum_{w < w(x^{\alpha_1} \wedge \dots \wedge x^{\alpha_q}) = w_0} f_w$$

lower weight terms

apply $g = g_{ij}(a)$. to f

$g f =$ H [H_1 (higher wt terms than w_0 coming from expansion of $g(x^{\alpha_1} \wedge \dots \wedge x^{\alpha_q})$.) these all have some power of a. some power of a

+ H_2 (higher wt terms from f_w, $w < w_0$ than w_0) larger a. these have higher power of a than]

+ (* other coeffs from f_w, $w < w_0$) also have a power of a

+ (lower wt terms)

think of $a \in k$ indeterminate.

since $\mathrm{gin}(I) \longleftrightarrow X^{\alpha_1} \wedge \cdots \wedge X^{\alpha_b}$

that the entire sum H is 0 $\forall a \in k$.

$\Rightarrow H_1 + H_2 = 0$

$\Rightarrow H_1 = 0$

$\Rightarrow \mathrm{gin}(I)$ is B-fixed.

<u>exercise</u> to flesh this out.

Properties of Borel-fixed ideals.

exercise given $J = (x^{\alpha_1}, \dots, x^{\alpha_r}) \subseteq S$
monomial ideal, B-fixed, char $k = 0$.
suppose $x^{\alpha_1} > \dots > x^{\alpha_r}$ ($x_0 > x_1 > \dots > x_n$, $>$ term order)
compute Hilbert Series, Function + Poly.
for S/J.

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recall (k = infinite field):

$(h_1, \dots, h_r)$, each $h_i \in S_1$

$(h_1, \dots, h_r)$ is a regular sequence on S/I if

$$S/I \xrightarrow{h_1} S/I$$

$$S/(I, h_1) \xrightarrow{h_2} S/(I, h_1)$$

$$\vdots$$

$$S/(I, h_1, \dots, h_{r-1}) \xrightarrow{h_r} S/(I, h_1, \dots, h_{r-1})$$

if these maps are all injective.

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def depth S/I = max length of such a linear regular sequence on S/I.

(similarly: replace S/I by M get depth M as well).

example $S = k[x_0, \ldots, x_4]$

$$J = (x_0^2, x_0x_1, x_0x_2, x_1^3)$$

associated primes of J: (x_0, x_1) (minimal prime)

(x_0, x_1, x_2)

depth S/J = ??

linear reg seq: (x_4, x_3) max reg. sequence

$\therefore$ depth $S/J = 2$.

[Auslander–Buchsbaum:

$$\mathrm{pd}_S\, S/J + \mathrm{depth}(S/J) = \dim S = n+1]$$

In this case: $\mathrm{pd}_S\, S/J = 5 - 2 = 3$.

exercise find the free resolution of S/J, check this!

Prop Suppose $J \subseteq S = k[x_0, \ldots, x_n]$ is B-fixed.

suppose x_ℓ divides a minimal gen. of J but $x_{\ell+1}$ does not.

then (a) $(x_n, x_{n-1}, \ldots, x_{\ell+1})$ is a max. regular sequence on S/J.

(b) $\operatorname{depth}(S/J) = n - \ell$

(c) $\operatorname{pd}_S(S/J) = \ell + 1$.

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next time: coherent sheaves on $\mathbb{P}^n \longleftrightarrow$ graded modules

discuss cohom. of sheaves, local duality

Castelnuovo-Mumford reg

(of modules, coherent sheaves)