

Sheaves + cohomology

Synopsis for the working joe.

Coherent sheaves on \mathbb{P}^n (or proj. subvarieties)
+ graded S -modules.

① Given: M f.g. graded S -module.

make a sheaf \tilde{M} on \mathbb{P}^n : $U_i = \mathbb{P}^n - V(x_i)$

$$\tilde{M}(U_i) := \left(M \otimes_S S\left[\frac{1}{x_i}\right] \right)_{\text{deg } 0 \text{ part}}$$

this is a module over

$$\mathcal{O}_{\mathbb{P}^n}(U_i) = S\left[\frac{1}{x_i}\right]_{\text{deg } 0 \text{ part.}}$$

\tilde{M} is a coherent sheaf on \mathbb{P}^n

aka \tilde{M} coherent $\mathcal{O}_{\mathbb{P}^n}$ -module.

② every coherent sheaf on \mathbb{P}^n arises as some \tilde{M} as above.

③ examples:

① $\tilde{S} = \mathcal{O}_{\mathbb{P}^n}$

$\because X \hookrightarrow \mathbb{P}^n$

② $\tilde{S}_X = \mathcal{O}_X$ (really $i_* \mathcal{O}_X$)

$$\textcircled{3} \quad \widetilde{S(d)} = \mathcal{O}_{\mathbb{P}^n}(d)$$

11 Sep 2020
Lecture #5

②

use

$$\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$$

$$\textcircled{4} \quad N_{X/\mathbb{P}^n} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\widetilde{I}_X, \mathcal{O}_X)$$

$$= \mathcal{H}om_S(I_X, S/I_X)$$

$$= \mathcal{H}om_S(I_X/I_X^2, S/I_X)$$

⑤ $K_X (= \omega_X) =$ canonical sheaf on X

$$K_X = \text{Ext}_S^{\text{codim } X}(S/I_X, S(-n-1))$$

⑥ if $D \subseteq X$ is an effective divisor

with ideal $J \subseteq R = S/I_X$.

then $\mathcal{O}_X(-D) = \widetilde{J}$ $J =$ graded S -module
 $J \subseteq R$

$$\mathcal{O}_X(D) = \widetilde{\text{Hom}_R(J, R)}$$

⑦ can also do Ω_X^1 , Ω_X^p .

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③

④ facts about \sim :

- \sim is an exact functor.

- $\tilde{M} \cong \tilde{N} \iff \exists m \in \mathbb{Z}$ s.t.

$M_{\geq m} \cong N_{\geq m}$ as S -modules.

↑
all elems of degree $\geq m$

examples

① twisted cubic: $C \subseteq \mathbb{P}^3$ ideal I .

$$0 \rightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} a & b \\ b & c \\ c & d \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} b^2 - c^2 & bc - ad & ac - b^2 \end{bmatrix}} S \rightarrow S/I \rightarrow 0$$

sheafify:

$$0 \rightarrow \mathcal{O}(-3)^2 \rightarrow \mathcal{O}(-2)^3 \rightarrow \mathcal{O} \rightarrow \mathcal{O}_C \rightarrow 0$$

is exact.

② Koszul complex. Let $V = S_1$ (k -vs.)

$$\begin{array}{ccccccc}
 & & & S(-1)^{n+1} & & & \\
 & & & \uparrow & & & \\
 & & \wedge^2 V \otimes S(-2) & \longrightarrow & V \otimes_k S(-1) & \longrightarrow & S \longrightarrow k \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & [x_0, \dots, x_n] & & \\
 & & \uparrow & & & & \\
 & \wedge^{n+1} V \otimes S(-n-1) & & & & & \\
 \uparrow & & & & & & \\
 0 & & & & & &
 \end{array}$$

is an exact sequence, resolving k .

$$k \cong \widetilde{S/(x_0, \dots, x_n)} = 0 \text{ sheaf}$$

sheafify:

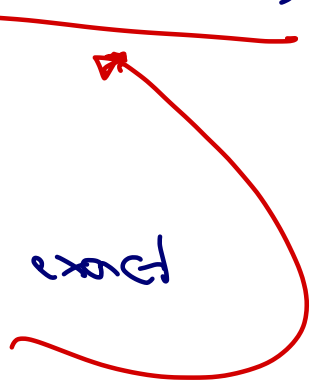
$$\begin{array}{ccccccc}
 & & & \mathcal{O}(-1)^{n+1} & & & \\
 & & & \uparrow & & & \\
 & & \wedge^2 V \otimes \mathcal{O}(-2) & \longrightarrow & V \otimes_k \mathcal{O}(-1) & \longrightarrow & \mathcal{O} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & & & \\
 & \wedge^{n+1} V \otimes \mathcal{O}(-n-1) & & & & & \\
 \uparrow & & & & & & \\
 0 & & & & & &
 \end{array}$$

is an exact sequence of locally free sheaves on \mathbb{P}^n .

If $F = \widetilde{M}$,

$\otimes_{\mathcal{O}_X} F$
 this is a complex

remains exact since



Get

$$\sigma \rightarrow \bigwedge^{m+1} V \otimes F(m-1) \rightarrow \dots \rightarrow V \otimes F(1) \rightarrow F \rightarrow 0$$

is exact.

(e) Given a coherent sheaf F on \mathbb{P}^n

define $H_*^0(F) = \bigoplus_{d \in \mathbb{Z}} H^0(F(d))$
finite dim'd k -vs.

get: $H_*^0(F)$ is a graded S -module.

most of the time, this is f.g!

if F has a component which is a point in \mathbb{P}^n

e.g: $F = \mathcal{O}_p$ $p \in \mathbb{P}^n$ is a point.

then $F = F(l)$ for all l . $H^0(\mathcal{O}_p) = k$

So $H_*^0(F) = \bigoplus_{d \in \mathbb{Z}} k$ not f.g.

not f.g

in all other cases, $H_*^0(F)$ is f.g.

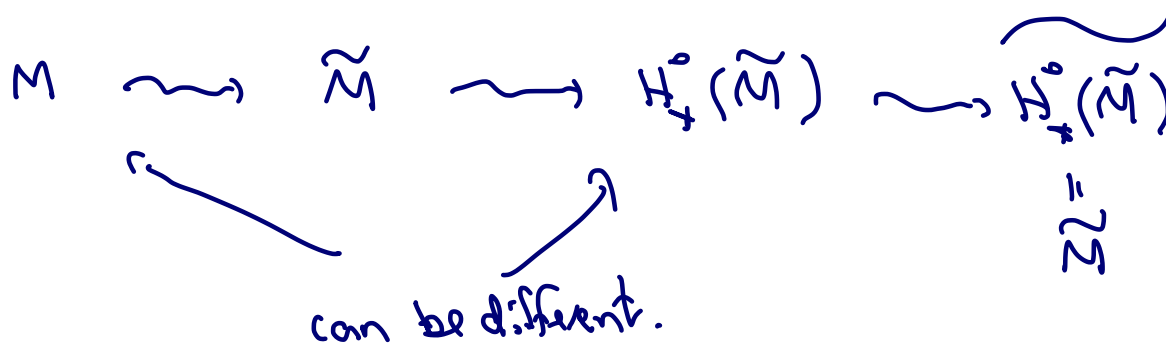
BUT note $H_*^0(F)_{\geq m}$ is f.g. $\forall m \in \mathbb{Z}$.

Ⓣ if $\tilde{\mathcal{I}}$ = ideal sheaf
of $X = V(\mathcal{I}) \subseteq \mathbb{P}^n$

then $H_*^0(\tilde{\mathcal{I}}) = \bigoplus_{d \in \mathbb{Z}} H^0(\tilde{\mathcal{I}}(d)) = \mathcal{I}^{\text{sat}}$

recall $\mathcal{I}^{\text{sat}} = (\mathcal{I} : (x_0, \dots, x_n)^\infty)$

(see Hartshorne
Ch. 2).



Coherent sheaf cohomology

Given F coherent sheaf on \mathbb{P}^n .

ⓐ Have $H^i(\mathbb{P}^n, F) = H^i(F)$ is a finite-dim^l
 k -vector space

ⓑ $H^i(F) = 0$ for $i < 0, i \geq n+1$.

© if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$
is an exact seq. of coherent sheaves on \mathbb{P}^n .

then

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^0(A) & \rightarrow & H^0(B) & \rightarrow & H^0(C) \\
 & & & & & & \searrow \\
 & & & & \rightarrow & H^1(A) & \rightarrow & H^1(B) & \rightarrow & \\
 & & & & & & \vdots & & & \\
 & & & & & & & & & \searrow \\
 & & & & \rightarrow & H^n(A) & \rightarrow & H^n(B) & \rightarrow & H^n(C) & \rightarrow & 0
 \end{array}$$

is exact sequence of k -vector spaces

④ Serre's theorem (also Cartan, Betti, Grothendieck)

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \cong S_d \quad d \in \mathbb{Z}$$

$$H^i(\mathcal{O}_{\mathbb{P}^n}(d)) = 0 \quad \text{for: } 1 \leq i \leq n-1 \text{ all } d \in \mathbb{Z}.$$

$$H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d)) \cong S'_d \quad \text{k-vector space dual of } S_d.$$

$\forall d.$

$$(H^n(\mathcal{O}_{\mathbb{P}^n}(-n)) = 0, H^n(\mathcal{O}_{\mathbb{P}^n}(m)) = 0 \quad m \geq -n).$$

② Some useful exact sequences

(collect these, trade with your friends)

starter pack:

$$\textcircled{1} \quad 0 \rightarrow I_X \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_X \rightarrow 0$$

(comes $0 \rightarrow I_X \rightarrow S \rightarrow S_{(I_X)} \rightarrow 0$)