Lecture #5

Sheaves + cohomology
Synopsis for the working joe.

$\boxed{\text{Coherent sheaves on } \mathbb{P}^n \text{ (or proj. subvarieties)} + \text{graded } S\text{-modules.}}$

ⓐ Given: M f.g. graded S-module.

make a sheaf $\tilde{M}$ on $\mathbb{P}^n$: $\quad U_i = \mathbb{P}^n \setminus V(x_i)$

$$\tilde{M}(U_i) := \left(M \otimes_S S\left[\tfrac{1}{x_i}\right]\right)_{\deg 0 \text{ part}}$$

this is a module over

$$\mathcal{O}_{\mathbb{P}^n}(U_i) = S\left[\tfrac{1}{x_i}\right]_{\deg 0 \text{ part.}}$$

$\tilde{M}$ is a coherent sheaf on $\mathbb{P}^n$
aka $\tilde{M}$ coherent $\mathcal{O}_{\mathbb{P}^n}$-module.

ⓑ every coherent sheaf on $\mathbb{P}^n$ arises as some $\tilde{M}$ as above.

ⓒ examples: ① $\tilde{S} = \mathcal{O}_{\mathbb{P}^n}$

$i : X \hookrightarrow \mathbb{P}^n$

② $\widetilde{S/I_X} = \mathcal{O}_X$ (really $i_*\mathcal{O}_X$).

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③ $\widetilde{S(d)} = \mathcal{O}_{\mathbb{P}^n}(d)$

use $\boxed{\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}}$

④ $N_{X/\mathbb{P}^n} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\tilde{I}_X, \mathcal{O}_X)$

$= \widetilde{\mathrm{Hom}_S(I_X, S/I_X)}$

$= \widetilde{\mathrm{Hom}_S(I_X/I_X^2, S/I_X)}$

⑤ $K_X \ (= \omega_X) =$ canonical sheaf on X

$K_X = \widetilde{\mathrm{Ext}_S^{\operatorname{codim} X}(S/I_X, S(-n-1))}$

⑥ if $D \subseteq X$ is an effective divisor with ideal $J \subseteq R = S/I_X$.

then $\mathcal{O}_X(-D) = \tilde{J}$ $\quad$ J = graded S-module $J \subseteq R$

$\mathcal{O}_X(D) = \widetilde{\mathrm{Hom}_R(J, R)}$

7. can also do Ω^1_X, Ω^p_X.

8. facts about $\sim$:

- $\sim$ is an <u>exact</u> functor.
- $\widetilde{M} \simeq \widetilde{N} \iff \exists\, m \in \mathbb{Z}$ s.t.

$M_{\geq m} \simeq N_{\geq m}$ as S-modules.

all elems of degree $\geq m$

<u>examples</u>

1. twisted cubic: $C \subseteq \mathbb{P}^3$ ideal I.

$$0 \to S(-3)^2 \xrightarrow{\begin{bmatrix} a & b \\ b & c \\ c & d \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} bd-c^2 & bc-ad & ac-b^2 \end{bmatrix}} S \to S/I \to 0$$

sheafify:

$$0 \to \mathcal{O}(-3)^2 \to \mathcal{O}(-2)^3 \to \mathcal{O} \to \mathcal{O}_C \to 0$$

is exact.

2. Koszul complex. Let $V = S_1$ (k-v.s.)

$$0 \to \Lambda^{n+1} V \otimes S(-n-1) \to \cdots \to \Lambda^2 V \otimes S(-2) \to \underset{\cong\, S(-1)^{n+1}}{V \otimes_k S(-1)} \to \underset{k[x_0, \dots, x_n]}{S} \to k \to 0$$

is an exact sequence, resolving k.

$\tilde{k} = \widetilde{S/(x_0, \dots, x_n)} = 0$ sheaf

<u>sheafify</u>:

$$0 \to \Lambda^{n+1} V \otimes \mathcal{O}(-n-1) \to \cdots \to \Lambda^2 V \otimes \mathcal{O}(-2) \to \underset{\cong\, \mathcal{O}(-1)^{n+1}}{V \otimes_k \mathcal{O}(-1)} \to \mathcal{O} \to 0$$

is an exact sequence of locally free sheaves on $\mathbb{P}^n$.

If $F = \tilde{M}$, this complex $- \otimes_{\mathcal{O}_{\mathbb{P}^n}} F$ remains exact since

Get

$$0 \to \Lambda^{n+1} V \otimes F(-n-1) \longrightarrow \quad \cdots \longrightarrow V \otimes F(-1) \to F \to 0$$

is exact.

(e) Given a coherent sheaf F on $\mathbb{P}^n$

define $$H^0_*(F) = \bigoplus_{d \in \mathbb{Z}} \underbrace{H^0(F(d))}_{\text{finite dim'l } k\text{-v.s.}}$$

get: $H^0_*(F)$ is a graded S-module.

most of the time, this is f.g!

if $\overset{\text{supp}}{F}$ has a component which is a point in $\mathbb{P}^n$

e.g: $F = \mathcal{O}_p$ $\quad p \in \mathbb{P}^n$ is a point.

then $F = F(\ell)$ for all ℓ. $\quad H^0(\mathcal{O}_p) = k$

So $H^0_*(F) = \bigoplus_{d \in \mathbb{Z}} k$ not f.g.

not f.g

in all other cases, $H^0_*(F)$ is f.g.

BUT note $H^0_*(F)_{\geq m}$ is f.g. $\forall\, m \in \mathbb{Z}$.

(f) if $\widetilde{I}$ = ideal sheaf of $X = \mathbb{V}(I) \subseteq \mathbb{P}^n$

then $H^0_*(\widetilde{I}) = \bigoplus_{d\in\mathbb{Z}} H^0(\widetilde{I}(d)) = I^{sat}$

recall $I^{sat} = (I : (x_0, \dots, x_n)^\infty)$ (see Hartshorne ch. 2).

$$\underbrace{M \rightsquigarrow \widetilde{M} \rightsquigarrow H^0_*(\widetilde{M})}_{\text{can be different.}} \rightsquigarrow \widetilde{H^0_*(\widetilde{M})} = \widetilde{M}$$

Coherent sheaf cohomology

Given F coherent sheaf on $\mathbb{P}^n$.

(a) Have $H^i(\mathbb{P}^n, F) = H^i(F)$ is a finite-dim'l k-vector space

(b) $H^i(F) = 0$ for $i < 0$, $i \geq n+1$.

(c) if $0 \to A \to B \to C \to 0$

is an exact seq. of coherent sheaves on $\mathbb{P}^n$.

then

$$0 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to \cdots \to H^n(A) \to H^n(B) \to H^n(C) \to 0$$

is exact. sequence of k - vector spaces

(d) Serre's theorem (also Cartan, Bott, Grothendieck)

$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \simeq S_d \qquad d \in \mathbb{Z}$

$H^i(\mathcal{O}_{\mathbb{P}^n}(d)) = 0 \qquad$ for: $1 \le i \le n-1$, all $d \in \mathbb{Z}$.

$H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d)) \simeq S_d' \qquad$ k-vector space dual of S_d. $\forall d$.

$(H^n(\mathcal{O}_{\mathbb{P}^n}(-n)) = 0, \; H^n(\mathcal{O}_{\mathbb{P}^n}(m)) = 0 \quad m \ge -n)$.

e) $\boxed{\text{Some useful exact sequences}}$

(collect these, trade with your friends)

starter pack:

1) $$0 \to I_X \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

(comes $0 \to I_X \to S \to S/I_X$)