today: sheaf cohom
regularity

Exact sequences

2 if $h \in S_1$ linear form $H = V(h) \subseteq \mathbb{P}^n$
h regular on S/I_X. $X \subseteq \mathbb{P}^n$

$$0 \to \mathcal{O}_X(-1) \xrightarrow{h} \mathcal{O}_X \longrightarrow \mathcal{O}_{H \cap X} \longrightarrow 0$$

corner

$$0 \longrightarrow S/I(-1) \xrightarrow{h} S/I \longrightarrow S/I,H \longrightarrow 0$$

3 suppose $D \subseteq X$ is an effective divisor

$$0 \to \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

$R = S/I_X$, $I_D \subseteq R$

$$0 \to \widetilde{I_D} \to \widetilde{R} \longrightarrow \widetilde{R/I_D} \to 0$$

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important theorems (mostly by Serre)

let F = coherent sheaf on $\mathbb{P}^n$ (or on $X \subseteq \mathbb{P}^n$ projective).

- for $l \gg 0$, $i > 0$, $H^i(F(l)) = 0$
- for $l \gg 0$, $F(l)$ is generated by global sections:

$$H^0(F(l)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow F(l) \longrightarrow 0$$

$\boxed{\text{Local duality}}$ (FAC, Serre 1955)

M = graded S-module

$\widetilde{M} = F$ = corresp. coherent sheaf

First we need a def: $M = \bigoplus_{d \in \mathbb{Z}} M_d$

let $M^\vee := \bigoplus_{d \in \mathbb{Z}} M^\vee_d$ $\qquad M^\vee_d = M_{-d}'$ ← k-vector space dual.

this is a graded S-module.

<u>example</u> $M = S = k[x,y]$.

$$M^\vee = k\left[\frac{1}{x}, \frac{1}{y}\right]$$

$$y \cdot \frac{1}{x} = 0$$

$$x \cdot \frac{1}{x} = 1.$$

note: $\operatorname{ann}(1 \text{ in } M^\vee) = (x,y)$

FAC, Serre 1955

① $\forall\, i \geq 1$ $\quad \boxed{H^i_*(\widetilde{M})} = \bigoplus_{d \in \mathbb{Z}} H^i(\widetilde{M}(d))$ $\qquad$ sheaf cohom

(S-module)

$$= \boxed{\operatorname{Ext}^{n-i}_S(M, S(-n-1))} \checkmark \qquad \text{Ext}$$

$$= \boxed{H^{i+1}_{m}(M)} \qquad \text{local cohom.}$$

$$H^i(\widetilde{M}) = \operatorname{Ext}^{n-i}_S(M, S)'_{-n-1}$$

② $\exists$ exact sequence:

$$\begin{array}{ccccccccc} 0 \to & H^0_{\mathfrak{m}}(M) & \to & M & \to & H^0_*(\widetilde{M}) & \to & H^1_{\mathfrak{m}}(M) & \to 0 \\ & \wr\wr & & & & & & \wr\wr & \\ & \mathrm{Ext}^{n+1}_S(M, S(-n-1)) & & & & & & \mathrm{Ext}^n_S(M, S(-n-1)) & \end{array}$$

$$H^0_{\mathfrak{m}}(M) = \{ m \in M : (x_0, \ldots, x_n)^N m = 0 \text{ for } N \gg 0 \}$$

remarks

① Ext^{n+1}, Ext^n and Ext^{n-i} are f.g. (graded) S-modules

$\therefore$ they are 0 in low enough degree.

$\therefore$ for large ℓ, $M_{\geq \ell} \longrightarrow H^0_*(\widetilde{M})_{\geq \ell}$

is an isomorphism.

and $H^i(\widetilde{M}(\ell)) = 0 \quad \forall \ \ell \gg 0$.

② if $M = H^0_*(\widetilde{M})$, then $\mathrm{pd}_S M \leq n-1$

and $\mathrm{depth}_{\mathfrak{m}} M \geq 2$.

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Castelnuovo–Mumford regularity

Mumford: books on curves on an alg surface attributes this notion to Castelnuovo.

Def Let F be a coherent sheaf on $\mathbb{P}^n$

F is called m-regular if

$$H^i(F(m-i)) = 0 \quad \text{for all } i \geq 1.$$

(are 0 for large enough m)

recall: $\exists\, l_0$ s.t. for $l \geq l_0$ $H^i(F(l)) = 0$ $\forall\, i \geq 1$, $l \geq l_0$.

Prop (Mumford). Suppose that F is m-regular.

then (a) F is $(m+1)$-regular

(b) $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes_k H^0(F(m)) \longrightarrow H^0(F(m+1))$ is surjective.

(c) $F(m)$ is generated by global sections

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<u>Def</u> $\operatorname{reg} F := \min\{ m \in \mathbb{Z} : F \text{ is } m\text{-regular}\}$

<u>example</u> $\mathcal{O}(-l) = \mathcal{O}_{\mathbb{P}^n}(-l)$.

when is this m-regular?

<u>soln</u> $H^i(\mathcal{O}(-l)(m-i)) = 0 \quad \forall i \geq 1$

$\iff H^i(\mathcal{O}(m-l-i)) = 0 \quad \forall i \geq 1$

$\iff H^n(\mathcal{O}(m-l-n)) = 0$

$\iff m-l-n \geq -n$

$\iff m \geq l$.

$\therefore \operatorname{reg}(\mathcal{O}(-l)) = l$.

F m-regular $\implies$ $F(-1)$ is $(m+1)$-reg.

$F(1)$ is $(m-1)$-reg.

<u>example</u> twisted cubic

$$0 \to \tilde{I}_c \to \mathcal{O} \to \mathcal{O}_c \to 0$$

$$0 \to \mathcal{O}(-3)^2 \to \mathcal{O}(-2)^3 \to \tilde{I}_c \to 0$$

$$0 \to \mathcal{O}(m-3)^2 \to \mathcal{O}(m-2)^3 \to \tilde{I}_C(m) \to 0$$

is also exact.

So: $H^1(\tilde{I}_C(m-1)) = 0$ for all m

$H^2(\tilde{I}_C(m-2)) = 0 \quad \forall\, m \geq 2$

$H^3(\tilde{I}_C(m-3)) = 0 \quad \forall\, m \geq 0.$

$\therefore$ $\operatorname{reg} \tilde{I}_C = 2.$

use 1st sequence: $\operatorname{reg} \mathcal{O}_C = \operatorname{reg} \tilde{I}_C - 1.$

$= 1$

Lemma! If $0 \to A \to B \to C \to 0$ is exact, A, B, C are coherent sheaves on $\mathbb{P}^n$.

suppose that A is $(m+1)$-regular

B is m-regular.

then C is also m-regular.

proof

want $H^i(C(m-i)) = 0$

$\forall\, i \geq 1$

LES:

of $0 \to A(m-i) \longrightarrow B(m-i) \longrightarrow C(m-i) \longrightarrow 0$

$$H^i(B(m-i)) \longrightarrow H^i(C(m-i)) \longrightarrow H^{i+1}(A(m-i))$$

$H^i(B(m-i)) = 0$ since B is m-reg

$H^{i+1}(A(m-i)) = H^{i+1}(A((m+1)-(i+1))) = 0$

Since A is $(m+1)$-reg

$\therefore = 0$

want to prove Mumford's theorem.

exercise: try this yourself

First, regularity of a graded S-module.

Def Suppose this is a minimal free res:

$$0 \to F_{n+1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

$$\cdots \qquad F_1 = \bigoplus_{j=1}^{b_1} S(-d_{1j}) \qquad F_0 = \bigoplus_{j=1}^{b_0} S(-d_{0j})$$

Let $d_i := \max(d_{ij})$

Say M is *m-regular* if $m \geq d_i - i \quad \forall\, i \geq 0$

$$\operatorname{reg}(M) := \max(d_i - i \,,\ \text{all } i)$$