Lecture #7

Math 7670
Fall 2020
Stillman

16 Sep 2020
Lecture #7

Regularity.

Def $\mathcal{F}$ coherent sheaf is m-regular if $H^i(\mathcal{F}(m-i)) = 0 \quad \forall\, i \geq 1$.

$\operatorname{reg} \mathcal{F} = \min\{ m : \mathcal{F} \text{ m-reg}\}$.

Prop (Mumford) Suppose $\mathcal{F}$ is m-regular
then
(a) $\mathcal{F}$ is $(m+1)$-regular
(b) $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes_k H^0(\mathcal{F}(m)) \longrightarrow H^0(\mathcal{F}(m+1)) \to 0$
(c) $\mathcal{F}(m)$ is generated by global sections

recall: $0 \to \underset{\substack{\uparrow \\ m+1\text{ reg}}}{A} \to \underset{\substack{\uparrow \\ m\text{-reg}}}{B} \to C \to 0 \implies C$ m-reg.

proof of prop

Consider the Koszul complex.

$$V = S_1 \otimes_k S = S^{n+1}$$

have

$$0 \to \Lambda^{n+1} V \otimes S(-n-1) \to \cdots \to \Lambda^2 V \otimes S(-2) \to V \otimes S(-1) \xrightarrow[(x_0 \ldots x_n)]{} S \to k \to 0$$

is exact seq. of graded S-modules

sheafify:

$$0 \to \Lambda^{n+1} V \otimes \mathcal{O}(-n-1) \to \cdots \to \Lambda^2 V \otimes \mathcal{O}(-2) \to V \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0$$

this is exact seq. of locally free $\mathcal{O}_{\mathbb{P}^n}$-modules

Now $- \otimes_{\mathcal{O}_{\mathbb{P}^n}} F$. this remains exact (why is this true?)

16 Sep 2020
Lecture #7

(3)

Have the exact sequence

$$0 \to \Lambda^{n+1} V \otimes F(-n-1) \to \cdots \to \Lambda^2 V \otimes F(-2) \to V \otimes F(-1) \to F \to 0$$

Let $K_i = \ker\left(\Lambda^i V \otimes F(-i) \longrightarrow \Lambda^{i-1} V \otimes F(-i+1)\right)$

then

$$0 \to \underbrace{\Lambda^{n+1} V \otimes F(-n-1)}_{\text{reg: } m+n+1} \to \underbrace{\Lambda^n V \otimes F(-n)}_{\text{reg } m+n} \to K_{n-1} \to 0 \quad \Rightarrow \text{reg } K_{n-1} = m+n$$

$$0 \to \underbrace{K_{n-1}}_{m+n \text{ reg}} \to \underbrace{\Lambda^{n-1} V \otimes F(-n+1)}_{m+n-1 \text{ reg}} \to K_{n-2} \to 0 \quad \Rightarrow m+n-1 \text{ reg.}$$

$$\vdots$$

$$0 \to K_2 \to \Lambda^2 V \otimes F(-2) \to K_1 \to 0$$

$$0 \to \underbrace{K_1}_{m+2 \text{ reg}} \to \underbrace{V \otimes F(-1)}_{m+1 \text{ reg}} \to F \to 0 \quad \therefore m+1 \text{ reg.}$$

given F is m-regular

$\therefore F(-\ell)$ is $(m+\ell)$-regular.

$\therefore K_{n-1}$ is $m+n$ reg.

K_{n-2} is $m+n-1$ reg

$\vdots$

K_1 is $m+2$ reg.

F is $m+1$ reg // (a).

What about (b).

16 Sep 2020
Lecture #7

$$0 \to K_1(m+1) \to V \otimes F(m) \to F(m+1) \to 0$$

LES in cohom:

$$V \otimes H^0(F(m))$$
$$\parallel$$
$$H^0(V \otimes F(m)) \longrightarrow H^0(F(m+1)) \longrightarrow H^1(K_1(m+1)) \quad \text{is exact}$$

$$H^1(K_1(m+1)) = H^1(K_1((m+2)-1)) = 0$$

since K_1 is $m+2$-reg. $\therefore$ (b) ✓

For (c): do this yourself. ✓

Recall: M is a graded S-module.

then let

$$0 \to F_{n+1} \to \cdots \to \underbrace{\oplus S(-d_{1j})}_{F_1} \longrightarrow \underbrace{\oplus S(-d_{0j})}_{F_0} \longrightarrow M \to 0$$

be a minimal free resolution of M.

16 Sep 2020
Lecture #7

(5)

Def let $d_i := \max\{d_{ij} : \text{all } j\}$

$\operatorname{reg}(M) := \max(d_i - i)$

M is m-regular if $m \geq d_i - i \quad \forall i$.

Prop Suppose $M = H^0_*(F)$ is f.g. (graded) S-module.

then M is m-regular $\iff$ F is m-regular

ie $\operatorname{reg}(M) = \operatorname{reg}(F) \quad (F = \widetilde{M})$.

proof start with

$$0 \to F_{n+1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

sheafify:

$$0 \to \widetilde{F}_{n+1} \to \cdots \to \widetilde{F}_1 \to \widetilde{F}_0 \to \widetilde{M} \to 0$$

m-reg.

$\widetilde{F}_0 = \bigoplus_{j=1}^{b_0} \mathcal{O}(-d_{0j})$, $\widetilde{M} = F$

$(m+1)$-reg

$m \geq d_1 - 1$

$\therefore m+1 \geq d_1$

$m \geq d_0$.

$\widetilde{F}_0$ is d_0-regular

$\widetilde{F}_1$ is d_1-regular

$\vdots$

$\widetilde{F}_{n+1}$ is d_{n+1}-regular.

16 Sep 2020
Lecture #7

6

want to show : $\tilde{M}$
M m-reg , then $\overset{\text{"}}{F}$ is m-reg

lemma (essentially from the last proof) :
If $0 \to F_{n+1} \to F_n \to \cdots \to F_0 \to F \to 0$
is an exact sequence of coherent sheaves on $\mathbb{P}^n$
and F_i is $(m+i)$-regular. for $i \geq 0$.
then F is m-regular.

proof from last proof : break into short exact sequences.

Apply this lemma:
$\Rightarrow F$ is m-regular.
$\therefore \operatorname{reg} F \leq \operatorname{reg} M$

Suppose F is m-regular, show M is m-reg
need to construct a free resolution of M with degrees "small enough".

16 Sep 2020
Lecture #7

$$0 \to K_1 \longrightarrow \bigoplus S(-d_{0j}) \longrightarrow M \to 0$$

$d_{0j} \leq m \ \forall j$. as M is generated in degrees $\leq m$.

[Mumford's thm : $\underset{M}{\overset{\shortparallel}{}} H^0_*(F) =$ generated in degrees $\leq m$.]

suppose we can show that $\widetilde{K_1}$ is $(m+1)$-regular

if so :

$$0 \to K_2 \to \bigoplus S(-d_{1j}) \to K_1 \to 0$$

if we can do the same: $\widetilde{K_2}$ $(m+2)$-regular.

show $d_{1j} \geq m+1$

$d_{2j} \geq m+2$

$\vdots$

$\therefore$ show M is m-regular.

shows : $\operatorname{reg} M \leq m$ $\therefore$ $\operatorname{reg} M = \operatorname{reg} F$.

$$0 \to K_1 \to \bigoplus S(-d_{0j}) \to M \to 0$$

$$0 \to \widetilde{K}_1 \to \bigoplus \mathcal{O}(-d_{0j}) \to F \to 0$$

this proof is almost done.

next time

exercise try doing the rest of this

exercise Suppose $I \subseteq S$ is a strongly stable monomial ideal, with a generator in degree m, no generators in higher degree.

$\operatorname{reg} \widetilde{I} = m$

$\operatorname{reg} I = m$