$\boxed{\text{Lecture \#8}}$

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Redo proof: $M = H^0_*(F)$

$\tilde{M} = F$

goal: F is m-regular $\Longrightarrow$ M is m-regular

Lemma If $0 \to A \to B \to C \to 0$ is an exact sequence of coherent sheaves on $\mathbb{P}^n$, and B is $(m+1)$-regular, and C is m-regular and $H^0(B(m)) \longrightarrow H^0(C(m)) \to 0$ is surjective then A is $(m+1)$-regular.

pf want $H^1(A(m)) = 0$

$H^i(A(m+1-i)) = 0 \quad \forall i \geq 2.$

LES:

$$H^0(B(m)) \longrightarrow H^0(C(m)) \longrightarrow H^1(A(m)) \longrightarrow \underset{\substack{\parallel \\ 0}}{H^1(B(m))}$$

surjective

want $=0$

since B $(m+1)$-reg.

this is also about the same

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construct a free res. of M:

- know from Mumford's theorem: all gens of $M = H^0_*(F)$ are in degree $\leq m$

let $F_0 = \bigoplus_{j=1}^{b_0} S(-d_{0j}) \longrightarrow M \longrightarrow 0$

$\therefore d_0 = \max(d_{0j}) \leq m.$

let K_1 = kernel

$$0 \to K_1 \to F_0 \to M \to 0$$

$$0 \to \tilde{K}_1 \to \tilde{F}_0 \to \tilde{M} \to 0$$

$\tilde{F}_0$ m-reg. $\tilde{M} = F$ m-reg.

lemma $\Rightarrow$ $\tilde{K}_1$ $(m+1)$-reg.

$$0 \to K_2 \to \bigoplus S(-d_{1j}) \to K_1 \to 0$$

$\therefore d_1 \leq m+1$

lemma $\Rightarrow$ K_2 $(m+2)$-reg.

continue.

$$0 \to F_{n+1} \to \cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

where $d_i \leq m+i$ for all $i = 0, \ldots, n+1$

$\therefore \operatorname{reg} M \leq m.$

<u>Question</u> If $\operatorname{reg} M = m$ (M = f.g. graded S-module)

define $\operatorname{hilbreg} M :=$

$$\min \{ d : P_M(e) = h_M(e) \quad \forall\, e \geq d \}$$

is this related to $\operatorname{reg} M$?

<u>note</u> $P_M(z) = \chi(F(z))$ $\qquad \tilde{M} = F$

$$= \sum_{i \geq 0} (-1)^i h^i(F(z))$$

note: $h^0(F(z)) = \dim H^0(\tilde{M}(z)) = \dim M_z$.

$\boxed{\underline{\text{Def}} \quad h^i(\mathcal{F}) := \dim_k H^i(\mathcal{F})}$

but if $m = \operatorname{reg} M$.

then $h^i(F(z)) = 0$ for $z \geq \operatorname{reg} M - 1$
for $i \geq 1$

$\therefore P_M(z) = h^0(F(z))$ for $z \geq \operatorname{reg} M - 1$

$\therefore \operatorname{hilbreg}(M) \leq \operatorname{reg} M - 1$.

Question: is this best possible?

Plan: parametrize all closed subschemes $X \subseteq \mathbb{P}^n$ with Hilbert polynomial $p(z)$.

Q1 Given a numerical polynomial $p(z)$ of degree $d \leq n-1$ ($\hookrightarrow p : \mathbb{Z} \to \mathbb{Z}$).

(eg: $p(z) = \binom{z-3}{2} + \binom{z+5}{3}$)

does $\exists$ a closed subscheme X s.t. $p = p_X$?

Suppose $X = \mathbb{V}(I)$

$p = P_X = P_{S/I} = HP$.

know: $\exists\, m \gg 0$ s.t. $p(m) = h_{S/I}(m) = \dim_k S_m/I_m$.

This m depends on I.

Q2 $\{ \text{hilbreg}(I) : \underset{I \text{ saturated}}{I} \text{ s.t. } P_{S/I} = p \}$

is this bounded?

Q3 $\{ \text{reg}\, I : \underset{I \text{ saturated}}{I} \text{ s.t. } P_{S/I} = p \}$

is this bounded?

Q4 we will see:

$\text{reg}(I) \leq \text{reg}(\text{in}_<(I))$ for any term order $<$.

$\therefore$ can restrict to monomial ideals in Q2, Q3.

Nice result of Diane Maclagan

theorem let Σ be a set of monomial ideals in S, s.t. if $I, J \in \Sigma$ then $I \not\subseteq J$. then Σ is finite.

Macaulay + Gotzmann theorems

Def Let $d > 0$ be an integer, let $c > 0$ integers. the d^{th} Macaulay representation of c is the unique representation

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_e}{e}$$

where $k_d > k_{d-1} > \cdots > k_e \geq e$.

define $$c^{\langle d \rangle} := \binom{k_d + 1}{d+1} + \binom{k_{d-1}+1}{(d-1)+1} + \cdots + \binom{k_e+1}{e+1}$$

Good references ① Green's Generic Initial Ideals (6 lectures in comm alg)

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2. Bruns–Herzog CM Rings
   2nd edition.

3. Peeva Graded Syzygies

example $c = 28$, $d = 3$

$\binom{7}{3} = 35$ $\binom{6}{3} = 20$ $\binom{5}{3} = 10$

$$28 = \underbrace{\binom{6}{3}}_{20} + \underbrace{\binom{4}{2}}_{6} + \underbrace{\binom{2}{1}}_{2}$$

$$28^{\langle 3 \rangle} = \binom{7}{4} + \binom{5}{3} + \binom{3}{2} = 35 + 10 + 3 = 48$$

$\boxed{\text{Macaulay's theorem}}$ If $h = h_{S/I}$ is the HF of S/I, then $\boxed{h(d+1) \leq h(d)^{\langle d \rangle}}$ $\forall d$.

example can $(1, 4, 2, 4, 1)$ be the HF of some S/I?
$h_0\ h_1\ \underline{h_2}\ h_3\ h_4$

$2^{\langle 2 \rangle}$

2-rep of 2 is $\binom{2}{2} + \binom{1}{1}$

$$\therefore 2^{\langle 2 \rangle} = \binom{3}{3} + \binom{2}{2} = 2.$$

$$\therefore h_3 \leq 2^{\langle 2 \rangle} = 2 \qquad \text{c!}$$
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This result, cool as it is, is opaque.

Def A monomial ideal $I \subseteq S$ is called a lex-segment in degree d, if I_d is spanned by the 1st $\dim I_d$ monomials of S_d in (descending) lex order $(x_0 > x_1 > \dots > x_n)$

I is a lex ideal if it is a lexsegment in degree d $\forall d \geq 0$.

example $I = (x_0, x_1^3, x_1^2 x_2^3) \subseteq S = k[x_0, \dots, x_3]$

check: is I a lex ideal

prop (Macaulay) If I is a lex segment in degree d, and no generators in degree $d+1$ then

$$h_{S/I}(d+1) = h_{S/I}(d)^{\langle d \rangle}.$$

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