Recall:

d^{th} Macaulay rep of c:

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_e}{e}$$

$k_d > k_{d-1} > \dots > k_e \geq e.$

unique.

$$c^{\langle d \rangle} = \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{(d-1)+1} + \dots \binom{k_e+1}{e+1}$$

Macaulay: if $h = h_{S/I}$

then $h(d+1) \leq h(d)^{\langle d \rangle}$ $\forall d.$

Lexsegment ideals

Prop (Macaulay): I lexseg in degree d, no gens in deg $d+1$

then $$h_{S/I}(d+1) = h_{S/I}(d)^{\langle d \rangle}$$

Corollary (Macaulay)

If $h = h_{S/I}$, then $\exists$ ideal

$L = \mathrm{Lex}(h) \subseteq S$ s.t.

(a) L is a lex ideal

(b) $h_{S/I} = h_{S/L}$.

that is: every actual Hilbert function $h_{S/I}$

is realized by a lex ideal.

exercise (1) L is a lex ideal $\Longrightarrow$ L is strongly stable

(2) L is a lex ideal $\Longrightarrow$ L^{sat} is also a lex ideal.

(3) L segment in degree d

$\Rightarrow$ $L:(x_0,\dots,x_n)$ is lexsegment in degree $d-1$.

Gotzmann's persistence theorems (1978)

If $I \subseteq S$ be homog, generated in degrees $\leq d$. If $h_{S/I}(d+1) = h_{S/I}(d)^{\langle d \rangle}$

then (a) $h_{S/I}(\ell+1) = h_{S/I}(\ell)^{\langle \ell \rangle}$ $\forall\ \ell \geq d$.

(b) I is d-regular.

$\boxed{\text{example}}$ Let I = homog. ideal

L = lex ideal w. same Hilb fcn.

$\therefore P_{S/I} = P_{S/L}$ Hilb polys are same

$\therefore \exists\ m \geq 0$ s.t. $h_{S/I}(\ell) = h_{S/L}(\ell)$ $\forall\ \ell \geq m$.

suppose
$$\begin{cases} h_I(m) = h_L(m) \\ h_I(m+1) = h_L(m+1) \\ I \text{ gen. in degrees} \leq m \end{cases}$$
true in both degs L is lex

then $P_I = P_L$.

Suppose $m \gg 0$

$$h_I(m) = h_L(m). = p(m)$$

$$\underbrace{I_m}_{\dim = p(m)} \subseteq S_m$$

Hilbert polynomials

Let $p(z)$ be a numerical polynomial (takes integer values at integers). of degree $d \leq n-1$

$X \subseteq \mathbb{P}^n$

$$p_X(z) = \sum_{i=0}^{n} (-1)^i h^i(\mathcal{O}_X(z))$$

<u>Question</u> What conditions on $p(z)$ insure that $p(z) = p_X(z)$, some $X \subseteq \mathbb{P}^n$ of dim d.?

Use Macaulay, Gotzmann.

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Suppose $I \subseteq S$

and we have an integer m

s.t. $h(m+1) = h(m)^{\langle m \rangle}$ $\qquad h = h_{S/I}$.

Gotzmann $\Rightarrow$ know $h(\ell)$ for $\ell \geq m$.

$$h(m) = \binom{b_0}{m} + \binom{b_1}{m-1} + \cdots + \binom{b_{m-e}}{e} = \sum_{j=0}^{m-e} \binom{b_j}{m-j}$$

where $b_0 > b_1 > \cdots > b_{m-e} \geq e > 0$

$$\therefore \; h(m+k) = \sum_{j=0}^{m-e} \binom{b_j + k}{m-j+k} \qquad \forall \, k \geq 0.$$

Let $z = m+k$
$k = z-m$

$$h(z) = \sum_{j=0}^{m-e} \binom{z-m+b_j}{z-j}$$

$$= \sum_{j=0}^{m-e} \binom{z-m+b_j}{b_j-m+j}$$

Let $\lambda_j := b_j - m + j + 1$ $\qquad \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{m-e} \geq 1$

$$\lambda_{m-e} = b_{m-e} - m + (m-e) + 1$$
$$= b_{m-e} - e + 1. \geq 1$$

ie:

$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-e})$

is an integer partition.

Let $r = m-e+1$

So

$$h(z) = \sum_{j=0}^{r-1} \binom{z+\lambda_j - j - 1}{\lambda_j - 1}$$

$$\lambda = (\lambda_0 \geq \cdots \geq \lambda_{r-1} \geq 1).$$

<u>example</u> twisted cubic $p(z) = 3z+1$ HP

d	0	1	2	3	4	5	6	...
$h(d)$	1	4	7	10	13	16	19	...
$p(d)$	1	4	7	10	13	16	19	...
$h(d)^{\langle d \rangle}$		10	11	15	16			

$S = k[a,b,c,d]$

is L = saturated lex ideal

$L = (a, b^4, b^3c)$ has this HP.

also $h(4) = 13 = \binom{5}{4} + \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$

$\qquad\qquad\qquad\qquad 5 \quad 4 \quad 3 \quad 1$

$4+k = z$

$$\therefore h(4+k) = \binom{5+k}{4+k} + \binom{4+k}{3+k} + \binom{3+k}{2+k} + 1$$

$$= \binom{z+1}{1} + \binom{z}{1} + \binom{z-1}{1} + 1$$

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$$h(z) = \sum_{j=0}^{r-1} \binom{z+\lambda_j-j-1}{\lambda_j-1}$$

$$\lambda = (\lambda_0 \geq \cdots \geq \lambda_{r-1} \geq 1).$$

$\lambda = (2,2,2,1)$.

<u>Def</u> (a) Given a partition $\lambda = (\lambda_0, \ldots, \lambda_{r-1})$

$(d+1 =) \lambda_0 \geq \cdots \geq \lambda_{r-1} \geq 1$

define

$$g_{gotzmann}(\lambda; z) = \sum_{j=0}^{r-1} \binom{z+\lambda_j-j-1}{\lambda_j-1}$$

has degree $\lambda_0 - 1$ $(= d)$.

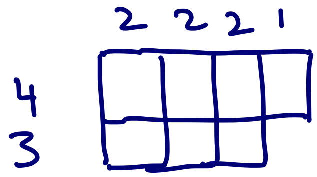
(b) Given a partition $\mu = (m_0, \ldots, m_d)$

$m_0 \geq m_1 \geq \cdots \geq m_d \geq 1$

define

$$g_{macaulay}(\mu; z) = \sum_{i=0}^{d} \left[\binom{z+i}{i+1} - \binom{z+i-m_i}{i+1} \right]$$

example $\lambda = (2,2,2,1)$



$D : (0,0)\quad (0,1)\quad (0,2)\quad (0,3)$
$\quad\ (1,0)\quad (1,1)\quad (1,2)$

$\lambda^c =: (4,3)$.

prop Let $\mu = \lambda^c$ $(\mu^c = \lambda)$.

then $g_{\text{macaulay}}(\mu ; z) = g_{\text{gotzmann}}(\lambda ; z)$

(Greg Smith, Andrew Staal).

proof use binomial identities:

given Young diagram D

$$g(D ; z) := \sum_{(i,j) \in D} \binom{z+i-j-1}{i}$$

show binomial identities imply equality //

<u>Prop</u> (Macaulay) Suppose $p(z)$ numerical poly, of degree $d \leq n-1$

TFAE

(a) $p = P_X$ for some $X \subseteq \mathbb{P}^n$ of dimension d.

(b) $\exists\ \mu = (m_0 \geq \dots \geq m_d \geq 1)$

s.t. $p = g_{mac}(\mu; z)$

(c) $\exists\ \lambda = (\lambda_0 \geq \dots \geq \lambda_{r-1} \geq 1)$ $\quad \lambda_0 = d+1$

s.t. $p = g_{gotzmann}(\lambda; z)$

in fact $\lambda = \mu^c$, $m_0 = r$

still need to see:

every $X \subseteq \mathbb{P}^n$ with this HP, has

$\mathrm{reg}(\tilde{I}_X) \leq m_0$.

e.g: twisted cubic:

every ideal w. same HP has $\mathrm{reg} \leq 4$

$\mu = (m_0 \geq m_1) = (4 \geq 3)$.