Hilbert Scheme lunch:

Wed. Sep 30 11:30 - 12:30

is this a good time to have it?

$w(b^2) = 2$

$w(ac) = 1$

$\boxed{\text{example}}$ $I = \langle \underline{b^2} - ac, \underline{bc} - ad, \underline{c^2} - bd \rangle$

$w = (0,1,1,0)$

$\mathrm{in}(I) = \langle b^2, bc, c^2 \rangle.$

Question: $I \underset{\text{flat family}}{\rightsquigarrow} \mathrm{in}(I)$

$$\begin{array}{c} X \subseteq \mathbb{A}^4 \times \mathbb{A}^1 \\ \pi \downarrow \\ \mathbb{A}^1 \end{array}$$

$\pi^{-1}(1) = V(I)$

$\pi^{-1}(0) = V(\mathrm{in}(I))$

consider

$$g_t = \begin{pmatrix} 1 & & & \\ & t^{-1} & & \\ & & t^{-1} & \\ & & & 1 \end{pmatrix} \in GL(4) \qquad t \neq 0.$$

$$\lim_{t \to 0} g_t(I) \overset{?}{=} \mathrm{in}(I).$$

$\underline{t \neq 0}$: $g_t I = \langle t^{-2}b^2 - t^{-1}ac, t^{-2}bc - ad, t^{-2}c^2 - t^{-1}bd \rangle$

$\underline{\underline{t \in \mathbb{C}}}$ $= \langle b^2 - tac, bc - t^2 ad, c^2 - tbd \rangle$

as $t \to 0$: $= \langle b^2, bc, c^2 \rangle.$

Situation:

$I \subset S = k[x_0, \dots, x_n]$ homog.

$>$ = term order.

Let $\vec{w} \in \mathbb{N}^{n+1}$

define the w-weight of x^a to be $\vec{w} \cdot \vec{a} \in \mathbb{N}$

$w(x^a) := w \cdot a$.

define $>_w$ by $x^a >_w x^b \iff w(a) > w(b)$

this is <u>not</u> a <u>total</u> order

Define $\mathrm{in}_w(f)$ = sum of the terms of f of maximal w-weight.

$\mathrm{in}_w(I) = k\text{-span}(\mathrm{in}_w(f) : f \in I) \subseteq S$ is an ideal.

If $G = \{g_1, \dots, g_r\} \subseteq I$ and

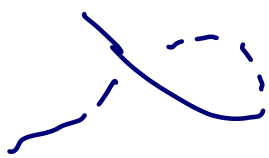
$\langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_r) \rangle = \mathrm{in}_w(I)$

call G a <u>GB wrt w</u> (or <u>w-GB</u>).

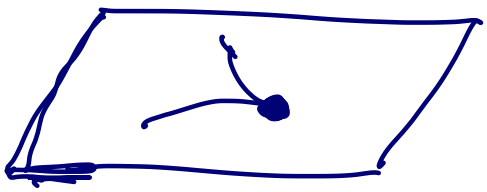
(often <u>not</u> a monomial ideal)

$\boxed{\text{example}}$ (flat projection)

I = twisted cubic



twisted cubic in $\mathbb{P}^3$



$c=0$ plane in $\mathbb{P}^3$

$c=0$
$b^3-a^2d=0$.

Let $w=(0,0,1,0)$
"c" component.

$$G=\{\underline{b^3-a^2d},\ \underline{bc}-ad,\ \underline{ac}-b^2,\ \underline{c^2}-bd\}$$

$$\mathrm{in}_w(G)=\langle b^3-a^2d,\ bc,\ ac,\ c^2\rangle = \mathrm{in}_w(I).$$

$$\sqrt{\mathrm{in}_w(G)}=\langle c,\ b^3-a^2d\rangle$$

$$\underline{\mathrm{in}_w(I)} = (c,\ b^3-a^2d)\cap\underbrace{(a,bc,c^2,b^3)}_{\text{embedded point}}$$

<u>Def</u> Let $w \in \mathbb{N}^{n+1}$, $>$ = term order

then <u>$>$ refines w</u> if whenever $x^a >_w x^b$

then $x^a > x^b$.

$\boxed{\text{simple prop}}$ If $>$ refines w, and $G = GB$ for I

wrt $>$, then $G = GB$ for I wrt w

and $in_w(G) = in_w(I)$.

$\boxed{\text{Lemma}}$ Suppose $x^{a_1} > x^{b_1}$, $x^{a_2} > x^{b_2}$, ..., $x^{a_N} > x^{b_N}$.

[i.e.: have $>$,
monoms x^{a_i}, x^{b_i}]

then $\exists\ w \in \mathbb{N}^{n+1}$ s.t. $w(a_1) > w(b_1)$

$\vdots$

$w(a_N) > w(b_N)$

(in fact, $\exists$ entire cone of such w's in $\mathbb{N}^{n+1}$).

$\boxed{\text{this implies:}}$ Fix $>$ term order, and a degree M.

then $\exists\ w \in \mathbb{N}^{n+1}$ s.t. $>_w$, $>$ agree on all

monomials of degree $\leq M$.

<u>example</u> flat projection

$$G = (b^3 - a^2 d,\ bc - ad,\ ac - b^2,\ c^2 - bd)$$

$w = (0,1,4,0)$,

w matches the order $c \gg b \gg (a,d)$ grevlex

in degrees ≤ 3.

$$\mathrm{in}_w(G) = \mathrm{in}_>(I) = \mathrm{in}_w(I) = \langle b^3, bc, ac, c^2 \rangle.$$

Construct the flat family $I \rightsquigarrow \mathrm{in}_w(I)$.

Homogenization:

fix $w \in \mathbb{N}^{n+1}$

let $\tilde{S} = k[t][x_0, \dots, x_n] = k[x_0, \dots, x_n, t]$.

for $f \in S$

$$f = c_1 x^{a_1} + c_2 x^{a_2} + \dots + c_\ell x^{a_\ell} \qquad w(a_1) \geq \dots \geq w(a_\ell)$$

let $b = \max\{w(a_i) : i = 1 \dots \ell\}$.

define $\tilde{f} := t^b f(t^{-w_0} x_0, \dots, t^{-w_n} x_n) \in \tilde{S}$

then $\tilde{f} = c_1 x^{a_1} + c_2 t^{b - w(a_2)} x^{a_2} + \dots$

for $I \subset S$ homog.

let $\tilde{I} := \langle \tilde{f} : f \in I \rangle \subseteq \tilde{S}$.

note $\tilde{I} \subseteq \tilde{S}$ is homog. under 2 degrees:

① $\deg(x_i) = 1$, $\deg(t) = 0$ ✓

② $\deg t = 1$, $\deg x_i = w_i$ ✓

Lemma Suppose $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_r) \rangle$

for $g_1, \ldots, g_r \in I$

then if $h \in I$, $\exists$ representation

$$h = \alpha_1 g_1 + \cdots + \alpha_r g_r \qquad \alpha_i \in S.$$

s.t. $\mathrm{in}_w(h) \geq_w \mathrm{in}_w(\alpha_i g_i) = \mathrm{in}_w(\alpha_i)\,\mathrm{in}_w(g_i)$

$\forall i = 1 \ldots r$, $=$ occurs at least once.

$\left(w(\mathrm{in}_w(h)) \geq w(\mathrm{in}_w(\alpha_i g_i)) \right)$

proof choose h of min. w-degree ($w(\mathrm{in}_w(h))$ minimal)

s.t. this fails.

$h \in I$, $\therefore$ $\mathrm{in}_w(h) \in \mathrm{in}_w(I)$

$\therefore$ $\exists\, \beta_1, \dots, \beta_r$ each either 0 or w-homog

s.t. $\mathrm{in}_w(h) = \beta_1 \underline{\mathrm{in}_w(g_1)} + \dots + \beta_r \underline{\mathrm{in}_w(g_r)}$

either $\beta_i = 0$ or $w(\mathrm{in}_w(h)) = w(\beta_i\, \mathrm{in}_w(g_i))$.

$h - \sum \beta_i g_i$ has lower wt. c!

$\therefore$ result holds. //

$\boxed{\text{corollary}}$ If $G = \{g_1, \dots, g_r\}$ is a w-GB of I

$(\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_r) \rangle)$

then (a) $\tilde{I} = \langle \tilde{g}_1, \dots, \tilde{g}_r \rangle$

(b) $\tilde{S}/\tilde{I}$ is flat over $k[t]$.

<u>note</u>: $\tilde{S}/\tilde{I} \otimes_{k[t]} k[t]/t-1 = S/I$

$\tilde{S}/\tilde{I} \otimes_{k[t]} k[t]/t = S/\mathrm{in}_w(I)$

(under identification: $\tilde{S} \otimes k[t]/(t-a) \simeq S$).

<u>proof</u>

(a) $\supseteq$ ✓

$\subseteq$

Suppose $\hat{h} \in \tilde{I}$. Note: if $h := \hat{h}(x,1)$ then $h \in I$, and $\underbrace{\hat{h} = t^M \tilde{h}}$ for some M.

If we show $\tilde{h} \in \tilde{I}$ we have proved (a) + (b) !

lemma $\Rightarrow$ $h = \sum \alpha_i g_i$ with $w(h) \geq w(\alpha_i g_i)$ equality occurs some i.

$$\therefore \tilde{h} = \sum t^{w(h) - w(\alpha_i g_i)} \tilde{\alpha}_i \tilde{g}_i \in \tilde{I}$$
$$\in \langle \tilde{g}_1, \ldots, \tilde{g}_r \rangle.$$

<u>corollary</u> $\exists$ flat family $I \rightsquigarrow \mathrm{in}_>(I)$ $\forall$ term order $>$.

next time:
- [flatness + free res's + HP's.]
- Functor of points, Hilbert schemes. Haiman-Sturmfels paper