Lecture #14

Haiman–Sturmfels

Some remarks:

- really want to consider:

  k = some base ring (e.g: $\mathbb{Z}$, field)

  = commutative ring (with 1).

Consider schemes over k.

(Spec $\mathbb{C}$, what is the functor of points?).

Usually consider for fixed k, X = k-scheme.

$\mathrm{Hom}_k(-, X)$

h_X : k-alg $\longrightarrow$ Sets
(comm!)

$R \longmapsto \mathrm{Hom}_k(\mathrm{Spec}\, R, X)$

$$\begin{tikzcd} \mathrm{Spec}\, R \arrow[rr] \arrow[dr] & \circledast & X \arrow[dl] \\ & \mathrm{Spec}\, k & \end{tikzcd}$$

as before h_X determines X uniquely.

- if $F, G \in \mathrm{Fun}(k\text{-alg}, \mathrm{sets})$

  $\alpha : G \longrightarrow F$: subfunctor

  $G(R) \hookrightarrow F(R)$

  [open subfunctor
  closed subfunctor
  (open) covering by open subfunctors.]

  [fiber products

$$\begin{array}{ccc} G \times_F H & \longrightarrow & G \\ \downarrow & & \downarrow \\ H & \longrightarrow & F \end{array}$$
]

  F "Zariski sheaf"

  (h_X is a Zariski sheaf).

Haiman–Sturmfels setup + main thms

k = commutative ring

$S = k[x_1, \ldots, x_n]$

$x^u \longleftrightarrow u \in \mathbb{N}^n$ monomials grading:

A = any Abelian group.

$\deg : \mathbb{N}^n \longrightarrow A$ semigroup homom.

$$S = \bigoplus_{a \in A} S_a \qquad S_a S_b \subseteq S_{a+b}.$$

$$S_a = k\text{-span}(x^u : \deg u = a)$$

($A = \mathbb{Z}$ our usual case, $\deg(x_i) = 1$)

Let $a_i := \deg(x_i) \in A$

WLOG: A is generated by $a_1, \dots, a_n$.

Let $A_+ := \deg(\mathbb{N}^n) \subseteq A$

Def $\deg$ is called positive if $S_0 = \text{span}_k(1)$.

(in this case $A_+ \cap (-A_+) = 0$).

Def A homog. ideal $I \subseteq S$ is admissible if

$(S/I)_a = S_a / I_a$ is a locally free k-module

of finite rank $\forall a \in A$ (constant on Spec k).

Its Hilbert function

$$h_I : A \longrightarrow \mathbb{N}$$
$$a \longmapsto \operatorname{rank}_k \left(S/I \right)_a .$$

<u>goal #1</u> Given $h : A \to \mathbb{N}$ supported on A_+ construct a k-scheme which parametrizes all admissible ideals $I \subset S$ with $h = h_I$

<u>functor of points</u>

Given S as above, k, $h : A \to \mathbb{N}$

define

$$H_S^k : k\text{-alg} \longrightarrow \text{sets}$$
$$R \longmapsto \{ I \subseteq R \otimes_k S : I \text{ homog}, \ R \otimes_k S_a / I_a \text{ is locally free of rank } h(a) \text{ over } R \}$$

define $H_S^k(f : R_1 \to R_2)$.

$\boxed{\text{Goal \#2}}$ (gives goal #1)

construct the k-scheme which represents this functor.

$\boxed{\text{theorem 1.1}}$

$\exists$ quasi-projective schem Z over k s.t.

$$\mathrm{Hom}_k(-, Z) \simeq H^h_S .$$

Denote this Z by Hilb^h_S.

$\boxed{\text{theorem 1.2}}$ If the grading is positive then Hilb^h_S is projective over k.

(in fact, will be contained in a finite product of Grassmannians).

$\boxed{\text{examples}}$

1. If $A = \mathbb{Z}$, $\deg(x_i) = 1$ $\forall i$. $\quad S = k[x_1, \dots, x_n]$

   $h = \mathbb{Z} \to \mathbb{N}$ a Hilbert function

   $$\mathrm{Hilb}^h_S = \{ I \subset S : S/I \text{ admissible}, HF = h \}$$

   this is a projective scheme over k.

$A = \mathbb{Z}$, $\deg(x_i) = 1\ \forall i$ $\quad S = k[x_0, \dots, x_n]$

(2) Given $p(z)$, HP for $\mathbb{P}^n$ with its m_0.

$$h(d) := \begin{cases} p(d) & d \geq m_0 \\ \binom{n+d}{n} & d < m_0 \end{cases}$$

note: $\underset{\subseteq S}{I}$ has HP $p(z)$, then

$I_{\geq m_0}$ has Hilbert function h.

$\therefore \ \mathrm{Hilb}_S^h = \mathrm{Hilb}^{p(z)}(\mathbb{P}^n)$.

(3) $A = \mathbb{Z}^2$ $\quad n = 3$ $\quad k[x, y, z] = S$ $\quad k = \text{field}$.

$\deg x = (1,0)$
$\deg y = (1,1)$
$\deg z = (0,1)$

Consider "9 points" in $\mathbb{A}^3$ $\quad (\dim_k S/I = 9)$

bivariate Hilbert series is:

$$s^2t^2 + s^2t + st^2 + s^2 + 2st + s + t + 1$$

exercise: construct all such "admissible" ideals

parametrize them: $\mathbb{P}^1 \cap \mathbb{P}^1 = \text{pt}$.

$\boxed{\text{Def}}$ (T, F) graded k-module with operators

k = comm ring

A = arbitrary set of indices, called "degrees"

$T = \bigoplus_{a \in A} T_a$ graded k-module

equipped with a collection of operators

$F = \bigcup_{a,b \in A} F_{ab}$ $\qquad F_{ab} \subseteq \mathrm{Hom}_k(T_a, T_b)$

assume F is closed under composition.

ie: (T, F) is a small category of k-modules

objects : T_a

Hom : elems of F.

for us : (S, F) $\qquad F_{ab}$ = set of monomials of degree $b-a$.

$S = \bigoplus_{a \in A} S_a$

(A Abelian gp).

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<u>exercise</u> Define, given $T = (T, F)$, h

$$H_T^h : k\text{-alg} \longrightarrow \text{Sets}$$

$$R \longmapsto H_T^h(R) = ??$$